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SUPERPOSITION AND APPROXIMATION IN RENWAL THEORY.(U)

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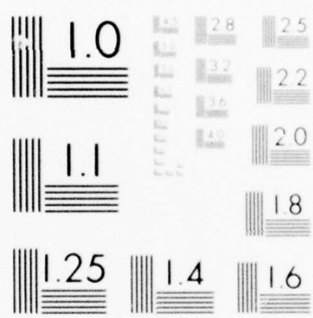

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SUPERPOSITION AND APPROXIMATION
IN RENEWAL THEORY

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ABSTRACT

This study is concerned with detailed expansions for the renewal function $H(t)$ and various of its generalizations. Our assumptions concerning the growth rate of the tail of the underlying lifetime distribution $F(x)$ are of the form $\int_0^\infty x^\nu M(x)\,dF(x) < \infty$, where $\nu \geq 0$ and $M(x)$ belongs to an appropriate class of monotone functions. When $F(x)$ has a finite variance, it is shown that (for $t > 0$)

$$H(t) = \frac{t}{\mu_1} - 1 + \frac{\mu_2}{2\mu_1^2} F_{(2)}(t) + L(t),$$

where μ_i denotes the ith moment of $F(x)$, $F_{(2)}(x)$ is the second derived distribution of $F(x)$, and $L(t)$ is a function of bounded variation such that, in particular, $L(t) = o(1/tM(t))$ as $t \to \infty$. A similar expression for $H(t)$ involving a finite number of derived distributions is derived for the infinite variance case. In addition our approach yields refined expansions for the factorial moments and cumulants of the number of renewals in the time interval $(0,t)$. By developing estimates for the moments of the forward recurrence-time we can also evaluate the variance of the number of renewals in an interval of time away from the origin.

Our main task throughout is to demonstrate that various remainder terms are functions $B(x)$ of bounded variation belonging to some moment class $\mathcal{B}(M;\nu)$ defined by the property $\int_{-\infty}^{\infty} |x|^\nu M(|x|)\,|dB(x)| < \infty$. For this purpose we prove an extension of a theorem due to Wiener, Pitt, Lévy, and Smith concerning analytic functions of Fourier-Stieltjes

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transforms of functions in $B(M; \nu)$; our version is the most general which can be obtained with respect to $M(x)$. We also demonstrate that certain convolutions which might be presumed to belong to $B(M; 0)$ are actually in $B(M; 1)$. In conjunction with the Wiener-Pitt-Lévy-Smith result this unexpected property (referred to as "smoothing magic") yields renewal theoretic results which are stronger than can be achieved using the former alone.

Several of these theorems are applied in a discussion of the time-dependent behavior of the superposition of identical independent renewal processes. Aspects considered include the distribution of the number of events in a time interval near the origin, as well as the variance-time and covariance-time functions.

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TABLE OF CONTENTS

CHAPTER I: INTRODUCTION

The purpose of this study is to develop detailed expansions for the renewal function and various of its generalizations. The renewal function (which we write as $H(t)$) is defined as follows: Suppose that $\{X_j\}_{j=1}^{\infty}$ is a sequence of positive iid random variables with distribution function $F(x)$. The random variables X_j may be regarded as the lifetimes of similar objects which are successively and instantaneously replaced; in other words, renewals take place at "times" $X_1, X_1+X_2, X_1+X_2+X_3, \ldots,$ and so on. Then N_t, the number of renewals by time t, is the largest integer k such that

$$X_1 + X_2 + \ldots + X_k < t,$$

and $H(t) = EN_t$ by definition. It is not difficult to show that

$$H(t) = \sum_{n=1}^{\infty} P\{X_1+X_2+ \ldots +X_n \le t\}.$$

The renewal function occurs in a variety of useful probability models and has been studied extensively by many authors. The most fundamental and earliest result concerning $H(t)$ is the Elementary Renewal Theorem which states that

$$\frac{H(t)}{t} \to \frac{1}{\mu_1} \quad \text{as} \quad t \to \infty,$$

where $\mu_1 = EX_j$ and the limit $1/\mu_1$ is interpreted as zero if $\mu_1 = +\infty$. The first rigorous proof of the Elementary Renewal Theorem

is due to Feller (1941) who made use of a Tauberian theorem for Laplace transforms.

The development of renewal theory since 1941 has proceeded along two parallel lines, depending on whether the random variables X_j are assumed to be lattice or non-lattice. Feller (1949) introduced the theory of recurrent events for the situation in which the random variables X_j are restricted to (say) the integers; a recurrent event E is said to occur at times $X_1, X_1+X_2, \ldots,$ and u_n is defined as the expected number of occurrences of E at time n. Erdös, Feller, and Pollard (1949) proved that

(1.0.1) $$u_n \to \frac{1}{\mu_1} \quad \text{as } n \to \infty$$

under an aperiodicity condition. Using (1.0.1) Feller (1949) showed that

(1.0.2) $$\sum_{j=0}^{n} \left(u_j - \frac{1}{\mu_1}\right) \to \frac{\mu_2 - \mu_1}{2\mu_1^2} \quad \text{as } n \to \infty,$$

provided $\mu_2 = EX_j^2 < \infty$. Various generalizations of (1.0.1) and (1.0.2) have been obtained since 1949, each involving some assumption about the sequence $f_n = P\{X_j = n\}$, $n = 0,1,2,\ldots$. These results are too numerous to review here; for a thorough discussion of the theory of recurrent events based on a unified approach we refer the reader to the study by Smith (1976).

The present work is concerned with the theory of renewals which differs from that of recurrent events only in that the random variables X_j are not restricted to a lattice of values. There u_n corresponds to the expected number of renewals in the time interval $(t, t+1]$; this

can be written as $H(t+1) - H(t)$ and is known as the Blackwell difference. Blackwell (1948) proved the following analogue of (1.0.1): For any fixed $\alpha > 0$,

$$H(t+\alpha) - H(t) \to \frac{\alpha}{\mu_1} \quad \text{as} \quad t \to \infty. \tag{1.0.3}$$

Later we shall refer to the following important generalization of (1.0.3):

<u>KEY RENEWAL THEOREM</u> (Smith, 1961; page 498) *Suppose that the distribution* $F(x)$ *is non-lattice and that* $k(x)$ *is Riemann-integrable in every finite interval and*

$$\sum_{n=-\infty}^{\infty} \max_{n<x\leq n+1} |k(x)| < \infty.$$

Then

$$\int_0^\infty k(x-z)dH(z) \to \frac{1}{\mu_1}\int_0^\infty k(z)dz$$

as $x \to \infty$*, where the right-hand side is interpreted as zero if* $\mu_1 = \infty$.

A consequence of the Key Renewal Theorem is the following extension of the Elementary Renewal Theorem:

<u>SECOND RENEWAL THEOREM</u> (Smith, 1954) *Suppose* $F(x)$ *is continuous and*

$$\mu_2 = \int_0^\infty x^2 dF(x) < \infty.$$

Then

$$(1.0.4) \qquad H(t) = \frac{t}{\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + o(1)$$

as $t \to \infty$.

The Second Renewal Theorem is a prototype of the results which we shall derive; our concern will be to develop approximate formulae for $H(t)$ (and several of its extensions) involving remainder terms which tend to zero as $t \to \infty$. The rate of convergence for such terms typically follows from the analytical properties imposed on $F(x)$, and we shall make two kinds of assumptions concerning the underlying distribution.

First, for technical reasons, it will be necessary to assume that $F(x)$ possesses a certain amount of smoothness, although certainly not as much as absolute continuity. In fact, we shall merely require that some iterated convolution of $F(x)$ possess an absolutely continuous component.

However our main assumptions will deal with the growth rate of the tail of the distribution $F(x)$. This rate can be expressed in various ways involving, for example, "o" or "O" terms of functions of slow growth. (A detailed discussion of such conditions in the recurrent events situation is given by Smith (1976).) Here we choose to measure growth rate by allowing for the existence of moments of a fairly general nature as follows:

$$(1.0.5) \qquad \int_0^\infty x^\nu M(x)\,dF(x) < \infty.$$

Note that (1.0.5) implies, in particular, that

$$1 - F(x) = o\left(\frac{1}{x^{\nu}M(x)}\right) \quad \text{as} \quad x \to \infty.$$

It will be necessary to restrict the choice of $M(x)$ to one of two classes of monotone functions; we refer to these families as $\mathcal{M}$ and $\mathcal{M}^*$. Roughly speaking, functions in $\mathcal{M}^*$ grow like polynomials. The class $\mathcal{M}$ contains $\mathcal{M}^*$ and is, in a sense, the most inclusive family with which we can deal. Apparently Smith (1967) introduced the systematic use of moment classes in renewal theory, and since then other authors have employed slightly different families of monotone functions.

Our two basic results concerning the renewal function are given in Chapters 2 and 4. The first (Theorem 2.4) provides an expansion for the renewal function in the *finite variance* case, i.e., when (1.0.5) can be assumed for $\nu \geq 2$ and $M(x) \in \mathcal{M}$. Theorem 2.4 sharpens the Second Renewal Theorem by replacing the remainder term "o(1)" with a known function and a new remainder term which is of the order $o(1/tM(t))$. The latter is, in fact, shown to be a function of bounded variation satisfying a condition similar to (1.0.5). The form of the expansion in Theorem 2.4 is not new; an ancestral version is disguised in the addendum to the paper by Smith (1967). However our result is slightly more general in terms of the moment class used and is actually the best which can be achieved via our particular approach.

Our second result (Theorem 4.6) concerning the renewal function deals with the situation where $F(x)$ possesses an *infinite variance* but satisfies (1.0.5) for $\nu = 1 + \delta$ with $0 < \delta < 1$. The expansion

for $H(x)$ in this case is far more complicated than in the previous one, and the analysis required is likewise more detailed. The main task in proving Theorem 4.6 is (again) to demonstrate that a certain remainder term is a function of bounded variation satisfying an integrability property like (1.0.5). Theorem 4.6 is the renewal theoretic analogue of a result due to Stone and Wainger (1967) and considerably generalizes subsequent work by Dubman (1970).

In Chapters 5 and 6 we extend our investigation to higher moments of the renewal counting process N_t. Chapter 5 is concerned with detailed expansions for the factorial moments

$$\phi_k(t) = E\left\{(N_t+1)(N_t+2) \;\ldots\; (N_t+k)\right\},$$

defined for $k = 1,2,3,\ldots$. These, in turn, yield information about the cumulants of N_t, thus refining results obtained by Smith (1959), as well as allowing for more general moment conditions. Our expressions for the second and third cumulants of N_t are presented in closed form, and we indicate an approach for dealing with higher order cumulants. However further work (perhaps of a combinatoric nature) will be required in order to evaluate in a direct fashion the constants involved in the higher order situations.

An interesting question related to the problem of finding the variance of N_t is that of evaluating the variance of the number of renewals in an interval of time *away from the origin*. This involves the moments of the forward recurrence-time ζ_t, which is defined as the time measured forward from t to the next renewal. Estimates for $E[\zeta_t^n]$ are discussed in Chapter 6, and specific results obtained for

$E[\zeta_t]$ and $E[\zeta_t^2]$ are used to study variance-time and covariance-time functions.

The mathematical tools which we use throughout our work are described in Section 2.1. Theorem 2.1 (due to Smith (1967)) provides detailed information about the remainder term in the Taylor expansion for characteristic functions. In order to deal with functions of characteristic functions Smith (1967) developed a modification of a well-known result due to Wiener, Lévy, and Pitt which, in its original form, states that the reciprocal of a non-vanishing function with an absolutely convergent Fourier series possesses an absolutely convergent Fourier series. Theorem 2.2, which is an extension of Smith's theorem, concerns analytic functions of functions in the class $\mathcal{B}^{\dagger}(M; \nu)$, which is defined as the Banach algebra of Fourier-Stieltjes transforms of functions $B(x)$ of bounded variation such that

$$\int_{-\infty}^{\infty} |x|^{\nu} M(|x|)\,|dB(x)| < \infty$$

for fixed $\nu \geq 0$ and $M(x) \in \mathcal{M}$. The proof of Theorem 2.2 (which is largely based on recent work done by Smith (1976)) is contained in the Appendix. The key element in both the proof and the application of Theorem 2.2 is a device referred to as a *smooth mutilator function*, which bears some resemblance to the test functions used in the theory of generalized distributions.

In order to establish certain results in renewal theory via the "Wiener-Pitt-Lévy-Smith approach" we must frequently demonstrate first that some Fourier-Stieltjes transform belongs to a particular class $\mathcal{B}^{\dagger}(M; \nu)$. Suppose, for instance, that $F(x)$ is a distribution function

satisfying (1.0.5) for $\nu = 2$ and some $M(x) \in \mathcal{M}$. Then its Fourier-Stieltjes transform (which we write as $F^{\ddagger}(\theta)$) belongs to $\mathcal{B}^{\ddagger}(M; 2)$. If we form the *first derived distribution* of $F(x)$, defined as

$$F_{(1)}(x) = \int_0^x \frac{1-F(u)}{\mu_1}\,du, \tag{1.0.6}$$

where μ_1 is the first moment of $F(x)$, then $F^{\ddagger}_{(1)}(\theta)$ is a member of $\mathcal{B}^{\ddagger}(M; 1)$ rather than $\mathcal{B}^{\ddagger}(M; 2)$. In transform notation (1.0.6) is written as

$$F^{\ddagger}_{(1)}(\theta) = \frac{1-F^{\ddagger}(\theta)}{-\mu_1 i\theta},$$

and we can interpret the "loss" of one whole moment as the price to be paid for dividing by $-i\theta$. (In fact, this effect is generally a consequence of applying the "integration operator" $1/(-i\theta)$ to a Fourier-Stieltjes transform.) Now suppose we convolve $U(x) - F_{(1)}(x)$ with itself, where $U(x)$ is the Heaviside unit function; clearly

$$[1 - F^{\ddagger}_{(1)}(\theta)]\,[1 - F^{\ddagger}_{(1)}(\theta)] \in \mathcal{B}^{\ddagger}(M; 1),$$

and one might naturally conclude in view of the above that

$$\frac{[1 - F^{\ddagger}_{(1)}(\theta)]^2}{-i\theta} \in \mathcal{B}^{\ddagger}(M; 0). \tag{1.0.7}$$

Surprisingly it can be shown (see Lemma 2.3) that the transform (1.0.7) belongs to the class $\mathcal{B}^{\ddagger}(M; 1)$; in other words, the convolution in the numerator of (1.0.7) has the unexpected effect of causing a "lost" moment to "reappear."

This property, which we refer to as *smoothing magic*, plays a vital role throughout Chapters 2, 4, 5, and 6. In conjunction with the Wiener-Pitt-Lévy-Smith theorem, smoothing magic yields renewal theoretic results which are much stronger than can be obtained by using Theorem 2.2 alone. Two extensions of the smoothing magic effect are discussed in Sections 4.3 and 5.2. We suspect, moreover, that these may themselves be special cases of some even more general property of convolutions such as (1.0.7).

An application of one of our main results (Theorem 2.4) is given in Chapter 3 which deals with an aspect of the time-dependent behavior of the superposition of identical independent renewal processes. This problem originally provided the motivation for studying detailed expansions for the renewal function. Suppose that the sources in the superposition consist of N renewal processes, each with underlying distribution $F(x)$ and corresponding first moment μ_1. Then the probability that no event in the (scaled) superposition occurs during $(t_0, t_0 + \Delta t]$ is given by

$$P_0 = \left\{1 - \int_{\mu_1 t_0/N}^{\mu_1(t_0+\Delta t)/N} \left[1 - F\left(\frac{t_0+\Delta t}{N/\mu_1} - u\right)\right] dH(u)\right\}^N .$$

Assuming N is small, if the superposition has not reached equilibrium by time t_0, then P_0 involves the behavior of the renewal function $H(x)$ for only moderately large values of x. By using Theorem 2.4 it is possible to obtain fairly sharp estimates for P_0 and related probabilities. The discussion of the superposition application is resumed in Chapters 5 and 6, where we derive approximations for the corresponding variance-time and covariance-time functions. The variance-

time curve has been used for statistical analysis of superposition (see Cox and Lewis (1966)), and it is conceivable that our work may lead to improved results in this direction.

CHAPTER II: SOME PRELIMINARY RENEWAL THEORY

The purpose of this chapter is to review some ideas and techniques related to the problem of characterizing the remainder term in a particular type of expansion for the renewal function. Our basic model is a sequence $\{X_n\}_{n=1}^{\infty}$ of iid random variables with distribution function $F(x)$ and corresponding renewal function defined as

$$H(x) = \sum_{n=1}^{\infty} P\{X_1 + \ldots + X_n \leq x\}.$$

For the applications we propose in Chapter 3, it is entirely reasonable to assume X_n is non-negative, although the discussion that follows below will extend to unrestricted random variables. In Section 2.2 we examine the asymptotic behavior of $H(x)$ as $x \to \infty$ *when* $F(x)$ *has a finite variance;* a study of the infinite variance situation will be taken up in Chapter 4.

2.1 Some Basic Tools

The notation and methods which we follow are largely based on a lengthy paper by Smith (1967) which appeared in the Proceedings of the Fifth Berkeley Symposium. The systematic approach developed in this work incorporates a number of features which are essential to our discussion in subsequent chapters.

Although a density is often assumed for $F(x)$ in applications, we shall obtain results under considerably weaker conditions. Consequently we shall refer to the following classes of distribution functions:

$\mathcal{T}$: distributions with a nonnull absolutely continuous component,

$\mathcal{C}$: distributions $F(x)$ such that, for some k, the kth iterated convolution of $F(x)$ is in $\mathcal{T}$,

$\mathcal{U}$: distributions $F(x)$ whose Fourier-Stieltjes transform $F^{\ddagger}(\theta)$ satisfies Cramér's Condition C: $\liminf_{|\theta| \to \infty} |1-F^{\ddagger}(\theta)| > 0$.

It is not difficult to show that $\mathcal{T} \subset \mathcal{C} \subset \mathcal{U}$.

The growth rate of the tail of the distribution $F(x)$ will play a critical role in our results. A key concept employed by Smith (1967) is the notion of a moment class of monotone functions. Smith introduced the class M^* of functions $M(x)$ satisfying the following conditions:

(1) $M(x)$ is nondecreasing in $[0,\infty)$,

(2) $M(x) \geq 1$ for all $x \geq 0$,

(3) $M(x+y) \leq M(x)M(y)$ for all $x, y \geq 0$,

(4) $M(2x) = O(M(x))$ for all $x \geq 0$.

$\mathcal{D}(M;\nu)$ will be used to denote the class of distribution functions $F(x)$ such that for some moment function $M(x)$ and some $\nu \geq 0$,

$$\int_{-\infty}^{\infty} |x|^{\nu} M(|x|)dF(x) < \infty .$$

A typical M^* function is $M(x)$ asymptotically equal to $x^{3/2}\log x$; a special M^* function is $I(x) \equiv 1$. Note that if $M(x) \in M^*$, then so is $x^{\alpha}M(x)$ where $\alpha \geq 0$. Condition (3) ensures that $\mathcal{D}(M;\nu)$ will be closed under convolusion. Note that Condition (4) excludes functions which grow exponentially fast, such as $M(x)$ asymptotically equal to $\exp\{x/(\log x)\}$ and $\exp\{x^{\delta}\}$ for $0 < \delta \leq 1$. In fact, if $M(x) \in M^*$ then $M(x) = O(x^{\beta})$ as $x \to \infty$ for some large β.

Various authors have introduced similar classes of monotone functions; see, for instance, Stone and Wainger (1967), Smith (1969), Essén (1973), Chover, Ney, and Wainger (1973), and Smith (1976). In a recent study of the theory of recurrent events Smith (1976) replaced Condition (4) with the less restrictive requirement

(4') For every fixed $h > 0$, $M(x+h) \sim M(x)$ as $x \to \infty$.

Functions in this class, which we shall denote by M, are referred to by Smith as *right moment functions*.

A function $T(x)$ (not necessarily in M) which satisfies (4') is said to be a *function of moderate growth* and has the canonical representation

$$(2.1.1) \qquad T(x) = b(x) \exp\left\{ \int_1^x \frac{\alpha(u)}{u} du \right\} \quad \text{as} \quad x \to \infty,$$

where $\alpha(u)/u \to 0$ as $u \to \infty$ and $b(x) \to 1$ as $x \to \infty$. Recall that functions of slow growth have the Karamata representation (2.1.1) where $\alpha(u) \to 0$ as $u \to \infty$; functions of regular variation are characterized by the requirement that $\alpha(u) \to \rho$ as $u \to \infty$, where ρ is positive and finite. $T(x)$ is a function of moderate growth iff $T(\log x)$ is a function of slow growth.

The advantage to be gained by adopting M rather than M^* is that we are able to include functions which grow asymptotically like $\exp\{x^{1/2}\}$ and $\exp\{x/(\log x)\}$. However our approach will necessitate what Smith (1976) has labelled the *Umbrella Condition U:*

$$\int_1^\infty \frac{|\log M(x)|}{1 + x^2} dx < \infty .$$

This excludes functions which grow asymptotically like e^x.

The membership requirements for M are essentially asymptotic. Smith (1969, 1976) pointed out that M may be extended to the class of functions $N(x)$ such that for some $M(x)$ satisfying (1) through (4') above, both $N(x)/M(x)$ and $M(x)/N(x)$ are bounded as $x \to \infty$.

Throughout our work we shall write $L(M; \nu)$ for the class of functions $f(x)$ such that for some $\nu \geq 0$ and $M(x)$ in some specified class,

$$\int_{-\infty}^{\infty} |x|^{\nu} M(|x|)|f(x)|dx < \infty.$$

Likewise $B(M; \nu)$ will denote the class of functions $B(x)$ of bounded variation satisfying

$$\int_{-\infty}^{\infty} |x|^{\nu} M(|x|)|dB(x)| < \infty.$$

Equivalently, functions in $B(M; \nu)$ are finite (complex) linear combi- *nations* of distributions functions in $D(M; \nu)$.

If $f(x)$ is in an L-class we write $f^{\dagger}(\theta) = \int_{-\infty}^{\infty} e^{i\theta x} f(x)dx$ for its Fourier transform. If $B(x)$ is in a B-class we write $B^{\ddagger}(\theta) = \int_{-\infty}^{\infty} e^{i\theta x} dB(x)$ to denote its Fourier-Stieltjes transform. This "dagger" notation will be applied in an obvious way to classes of functions; for example, the class of characteristic functions of distributions in $D(M; \nu)$ will be written as $D^{\ddagger}(M; \nu)$.

A main feature of the Smith (1967) approach is a theorem which provides detailed information about the remainder term in the Taylor expansion for a characteristic function. This result is adumbrated by a similar result of Smith (1959) concerning the Taylor expansion for the Laplace-Stieltjes transform of a distribution function, and it has found applications elsewhere in probability theory. Because we shall need to refer to the later result, the relevant portion is reproduced here:

<u>THEOREM 2.1</u> (Smith, 1967; page 270) *Let* $F(x) \in \mathcal{D}(I;\ \ell)$ *for some* $\ell > 0$, *and let* $k \geq 0$ *be the greatest integer not exceeding* ℓ.

(a) *When* ℓ *is not an integer, we can choose any real constant* c *and have*

$$F^{\dagger}(\theta) = 1 + \sum_{j=1}^{k} \frac{\mu_j}{j!} (i\theta)^j + |\theta|^{\ell} e^{ic \operatorname{sgn} \theta} s^{\dagger}(\theta)$$

where μ_j *is the* j^{th} *moment of* $F(x)$ and $s^{\dagger}(\theta) \in L^{\dagger}(I;\ 0)$.

(b) *If* $\ell \geq 1$ *and* r *is any integer,* $1 \leq r \leq \ell$, *for any* $F^{\dagger}(\theta) \in \mathcal{D}^{\dagger}(M;\ \ell)$ *we have*

$$F^{\dagger}(\theta) = 1 + \sum_{j=1}^{r-1} \frac{\mu_j}{j!} (i\theta)^j + \frac{(i\theta)^r}{r!} t_r^{\dagger}(\theta)$$

where $t_r^{\dagger}(\theta) \in \mathcal{B}^{\dagger}(M;\ \ell-r)$ *is the Fourier transform of some function* $t_r(x) \in L(I;0)$ *such that* $t_r^{\dagger}(0) = \mu_r$ *and*

$$\int_{-\infty}^{\infty} |t_r(x)|dx = \int_{-\infty}^{\infty} |x|^r \, dF(x).$$

When r *is even, or if* r *is odd and* $F(x)$ *refers to a non-negative random variable,* $t_r^{\dagger}(\theta) = \mu_r F_{(r)}^{\dagger}(\theta)$, *where* $F_{(r)}^{\dagger}(\theta) \in \mathcal{D}^{\dagger}(M;\ \ell-r)$. *In any case,* $t_r^{\dagger}(\theta)$ *is expressible as the linear combination of two characteristic functions, one of which corresponds to a positive random variable and the other to a negative one.*

We note that the original proof of Theorem 2.1 given by Smith (1967) uses only conditions (1) through (3) for $M(x) \in M^*$ and consequently suffices for $M(x) \in M$.

The distribution function $F_{(r)}(x)$ referred to in Theorem 2.1 is

known as the r^{th} *derived distribution* and can be written explicitly in terms of $F(x)$. A crucial consequence of Theorem 2.1 is that r whole moments are "lost" in going from $F(x)$ to $F_{(r)}(x)$. $F_{(r)}(x)$ is absolutely continuous, and we shall write $f_{(r)}(x)$ for the corresponding r^{th} derived density. Derived distributions are strikingly apposite to renewal theory; they were first used by Smith in his 1953 Cambridge Ph.D. dissertation, and Theorem 2.1 was foreshadowed in results obtained by Smith (1959).

Authors have typically analyzed the behavior of the renewal function by assuming a particular moment condition for $F(x)$ and deriving an expression for $H(x)$ which involves a remainder term of the "o" or "O" type or a function of slow growth. The methods employed are generally *ad hoc*. A well known example is the Second Renewal Theorem which states that if $F(x)$ is continuous and has a finite variance, then

$$H(x) = \frac{x}{\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1\right) + o(1) \quad \text{as} \quad x \to \infty.$$

Following the systematic approach developed by Smith (1967), we shall be concerned with proving that remainder terms belong to specific $\mathcal{B}$-classes. A number of results, including the more familiar kinds of estimates, can then be deduced from this stronger type of conclusion.

The cornerstone of this approach is a variation of a well-known theorem due to Wiener, Pitt, and Lévy. In its simplest form the original result states that the reciprocal of a nonvanishing function with an absolutely convergent Fourier series possesses an absolutely convergent Fourier series; see page 91 of Wiener (1933). Smith (1967) sharpened the Wiener-Pitt-Lévy theorem in order to deal with functions of Fourier-Stieltjes transform of functions of bounded variation. Following a

functional analytic approach Chover, Ney, and Wainger (1973) modified the Wiener-Pitt-Lévy theorem to obtain results with applications in the theory of branching processes.

Both these modifications are included in a very comprehensive version of the Wiener-Pitt-Lévy result developed by Smith (1976) for Fourier series. Much of our work will depend on the Fourier-Stieltjes analogue of a portion of Theorem 3.1 of Smith (1976). Since absolute continuity will be an issue, some additional notation is required: If $\phi(\theta)$ is a characteristic function, write $1 - \partial[\phi(\theta)]$ for the total weight of probability in the absolutely continuous component of the corresponding distribution. Define

$$\rho[\phi(\theta)] = \inf_k \{\partial[\{\phi(\theta)\}^k]\}^{1/k} .$$

In particular $\rho[\phi(\theta)] < 1$ if $\phi(\theta) \in C^{\ddagger}$.

We now state the version of the Wiener-Pitt-Lévy theorem which will be invoked later.

Theorem 2.2 (Wiener-Pitt-Lévy-Smith) *Suppose that* $\phi(\theta)$ *and* $\psi(\theta)$ *belong to* $B^{\ddagger}(M; \nu)$ *for some* $M(x) \in M$ *and some* $\nu > 0$, *and assume that* $\psi(\theta)$ *vanishes identically outside an interval* J. *Furthermore, suppose that as* θ *runs through* J, *the point* $z = \phi(\theta)$ *maps out an arc* C *in the complex plane and that* $\Phi(z)$ *is analytic at every point of* C.

(A) *If* J *is a compact interval then* $\psi(\theta)\Phi(\phi(\theta)) \in B^{\ddagger}(M; \nu)$.

(B) *If* J *is an infinite or semi-infinite interval and* $\psi(\theta) \in \mathcal{D}^{\ddagger}(M; \nu)$, then $\psi(\theta)\Phi(\phi(\theta)) \in B^{\ddagger}(M; \nu)$, *provided no singularity of* $\Phi(z)$ *is within a distance* $\rho[\phi(\theta)]$ *of the origin.*

The proof of Theorem 2.2 is essentially contained in the two papers by Smith (1967, 1976), obviating a detailed discussion at this point. Theorem 2.2 differs from Theorem 2 of Smith (1967) only in that we have replaced M^* by the broader class M. The (non-trivial) changes in the proof required by this generalization are given for the Fourier series situation by Smith (1976), and since the task of carrying out these modifications for the Fourier-Stieltjes case is lengthy albeit straightforward, we refer the reader to the Appendix for a complete proof of our version of the Wiener-Pitt-Lévy-Smith result.

One important modification does, however, merit special comment. Both proofs given by Smith (1967, 1976) involve the construction of a *smooth mutilator function*, abbreviated SMF. Given four real constants $\alpha < \beta < \gamma < \delta$ the SMF $q^{\dagger}(x;\ \alpha,\beta,\gamma,\delta)$ based on these points has the following properties: It vanishes when $x \le \alpha$ or when $x \ge \delta$ and has the value unity on the interval $\beta \le x \le \gamma$. It is monotonically increasing on $\alpha < x < \beta$ and monotonically decreasing on $\gamma < x < \delta$. The SMF is infinitely differentiable, and each derivative is bounded, vanishing identically, except when $\alpha < x < \beta$ or $\gamma < x < \delta$. Write

$$q(x;\ \alpha,\beta,\gamma,\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\, q^{\dagger}(\theta;\ \alpha,\beta,\gamma,\delta)\,d\theta.$$

The SMF constructed by Smith (1967) has the property that $q(x;\ \alpha,\beta,\gamma,\delta) \in L(M;\ \nu)$ for any $M(x)$ and any $\nu \ge 0$. However when M^* is replaced by M a much smoother SMF is required; for a construction see pages 47-48 of the paper by Smith (1976). We shall refer to this particular SMF in subsequent applications of Theorem 2.2.

It is interesting to note that the amount of smoothness that can be built into a SMF is restricted by the fact that it has a compact support.

Smith (1971) proved that a necessary and sufficient condition for the existence of a probability density function $p(x)$ with compact support such that $p^{\dagger}(\theta) \in L^{\dagger}(M; 1)$ is that $M(x)$ satisfy the Umbrella Condition; this is a consequence of Theorem XII of Paley and Wiener (1934). Since the construction of such a density is the essential step in the derivation of a SMF, we cannot hope to improve Theorem 2.2 to include functions $M(x)$ which grow faster than the rate permitted by the Umbrella Condition. Of course when $F(x) \in \mathcal{D}(M, \nu)$ and $M(x)$ grows like e^{x}, it is still possible to deal with renewal theoretic questions (and this has been done by various authors), but this situation requires the use of *ad hoc* techniques.

2.2 Smoothing Magic

We now use the Wiener-Pitt-Lévy-Smith approach to prove a result (Theorem 2.4) which describes the asymptotic behavior of the renewal function *when* $F(x)$ *has a finite variance.* Although Theorem 2.4 is foreshadowed in the addendum to the paper by Smith (1967), several novel features will emerge from our proof, as well as a slight generalization

of the preliminary version.

Assume that $F(x) \in \mathcal{D}(M; 2) \cap C$ for some $M(x) \in \mathcal{M}$. Since the renewal function $H(x)$ is not a function of totally bounded variation, it does not possess a Fourier-Stieltjes transform. When dealing with positive random variables corresponding to renewal lifetimes, this difficulty can be avoided by using Laplace-Stieltjes transforms. However we would then be forced to recast Theorems 2.1 and 2.2, thus losing the capability for possible future extensions to unrestricted random variables.

Instead we shall deal with a modified renewal function $H_\zeta(x)$ which is bounded, nondecreasing and absolutely continuous. Let $\Delta_a(x)$ be the triangular density function

$$\begin{aligned} \Delta_a(x) &= \frac{1}{a} - \frac{|x|}{a^2}, &\quad |x| \le a \\ &= 0, &\quad \textit{otherwise}, \end{aligned}$$

so that $\Delta_a^\dagger(\theta) = \dfrac{\sin^2(a\theta/2)}{(a\theta/2)^2}$. For $0 < \zeta < 1$ define (suppressing the dependence on a)

$$H_\zeta(x) = \sum_{n=0}^{\infty} \zeta^n \int_{-\infty}^{\infty} \Delta_a(x-z)\,dF_n(z), \tag{2.2.1}$$

where $F_o(x) = P\{0 \le x\}$ and $F_n(x) = P\{X_1 + \ldots + X_n \le x\}$, $n = 1,2,\ldots$.

Then

$$H_\zeta^\dagger(\theta) = \frac{\Delta_a^\dagger(\theta)}{1-\zeta F^\dagger(\theta)}.$$

By Theorem 2.1

$$F^\dagger(\theta) = 1 + \mu_1\, i\theta + \frac{\mu_2}{2}\,(i\theta)^2\, F^\dagger_{(2)}(\theta),$$

so that if we write $\beta = (1-\zeta) - \zeta\mu_1\, i\theta$, then

$$(2.2.2) \qquad H_\zeta^\dagger(\theta) = \frac{\Delta_a^\dagger(\theta)}{\beta + \frac{\mu_2}{2}\theta^2\zeta F_{(2)}^{\ddagger}(\theta)} .$$

Therefore

$$H_\zeta^\dagger(\theta) - \frac{\Delta_a^\dagger(\theta)}{\beta} = \frac{-\frac{\mu_2}{2}\theta^2\zeta\,\Delta_a^\dagger(\theta)F_{(2)}^\dagger(\theta)}{\beta[\beta + \frac{\mu_2}{2}\theta^2\zeta F_{(2)}^{\ddagger}(\theta)]} .$$

Let I be a small open interval containing the origin. For $\theta \in I$ it is not difficult to show that

$$|\beta + \frac{\mu_2}{2}\theta^2\zeta F_{(2)}^{\ddagger}(\theta)| > \frac{\mu_1}{2}\zeta|\theta| ,$$

and for all θ, $|\beta| > \mu_1\zeta|\theta|$. Consequently for $\theta \in I$

$$(2.2.3) \qquad |H_\zeta^\dagger(\theta) - \frac{\Delta_a^\dagger(\theta)}{\beta}| = O(1) ,$$

uniformly with respect to ζ. Write $\bar{H}_\zeta^\dagger(\theta)$ for the Fourier transform of the modified renewal function corresponding to the negative exponential distribution with mean μ_1. Clearly

$$\bar{H}_\zeta^\dagger(\theta) = \frac{\Delta_a^\dagger(\theta)(1 - \mu_1 i\theta)}{1 - \mu_1 i\theta - \zeta} ,$$

and for $\theta \in I$, there exists a positive constant C such that

$$(2.2.4) \qquad \begin{aligned} |\Delta_a^\dagger(\theta)|\,\left|\frac{1}{\beta} - \frac{1-\mu_1 i\theta}{1-\mu_1 i\theta-\zeta}\right| &\le C\left|\frac{\mu_1^2\zeta\theta^2}{\beta(1-\mu_1 i\theta-\zeta)}\right| \\ &\le \frac{C\mu_1|\theta|}{|1-\mu_1 i\theta-\zeta|} = \frac{C\mu_1|\theta|}{\sqrt{(1-\zeta)^2 + \mu_1^2\theta^2}} \le C. \end{aligned}$$

(2.2.3) and (2.2.4), together with the Triangle Inequality, imply that, for $\theta \in I$,

$$|H_\zeta^\dagger(\theta) - \overline{H}_\zeta^\dagger(\theta)| = O(1)$$

uniformly for $0 < \zeta < 1$.

Since we are assuming that $F^\ddagger(\theta)$ belongs to $C^\ddagger$, it follows that $\sup_{\theta \notin I} |F^\ddagger(\theta)| < 1$. This implies that $|H_\zeta^\dagger(\theta) - \overline{H}_\zeta^\dagger(\theta)|$ is bounded as $\zeta \uparrow 1$ for $\theta \notin I$. Furthermore (2.2.2), (along with the fact that $\Delta_a^\dagger(\theta)$ is integrable,) can be used to show that $H_\zeta^\dagger(\theta)$ is integrable for $\theta \notin I$, and, of course, the same claim may then be made for $\overline{H}_\zeta^\dagger(\theta)$.

Consequently we may legitimately apply the Fourier inversion formula

$$(2.2.5) \qquad H_\zeta(x) - \overline{H}_\zeta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\{H_\zeta^\dagger(\theta) - \overline{H}_\zeta^\dagger(\theta)\}d\theta,$$

since the integral in (2.2.5) is absolutely convergent. Setting

$$H(x) = \lim_{\zeta\uparrow 1} H_\zeta(x) \quad \text{and} \quad \overline{H}(x) = \lim_{\zeta\uparrow 1} \overline{H}_\zeta(x),$$

it follows by bounded convergence that

$$(2.2.6) \qquad H(x) - \overline{H}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\,\Delta_a^\dagger(\theta)\left\{\frac{1}{1-F^\ddagger(\theta)} - \left[\frac{1}{-\mu_1 i\theta} + 1\right]\right\}d\theta.$$

Using the expansion $F^\ddagger(\theta) = 1 + \mu_1 i\theta\, F_{(1)}^\ddagger(\theta)$ which is a consequence of Theorem 2.1, we may *formally* write

$$\frac{1}{1-F^\ddagger(\theta)} = \frac{1}{-\mu_1 i\theta F_{(1)}^\ddagger(\theta)} = \frac{1}{-\mu_1 i\theta}\left\{\frac{1}{1-[1-F_{(1)}^\ddagger(\theta)]}\right\}$$

$$= \frac{1}{-\mu_1 i\theta} \sum_{k=0}^{\infty} [1-F^{\ddagger}_{(1)}(\theta)]^k$$

$$(2.2.7) \qquad = \frac{1}{-\mu_1 i\theta} + \frac{1-F^{\ddagger}_{(1)}(\theta)}{-\mu_1 i\theta} + \frac{[1-F^{\ddagger}_{(1)}(\theta)]^2}{1 - F^{\ddagger}(\theta)} .$$

Therefore

$$\frac{1}{1-F^{\ddagger}(\theta)} - \frac{1}{-\mu_1 i\theta} = \frac{\mu_2}{2\mu_1^2} F^{\ddagger}_{(2)}(\theta) + L^{\ddagger}(\theta),$$

say, where $L^{\ddagger}(\theta) = \dfrac{[1-F^{\ddagger}_{(1)}(\theta)]^2}{1 - F^{\ddagger}(\theta)}$.

Our objective is to show that $L^{\ddagger}(\theta)$ is the Fourier-Stieltjes transform of a function of bounded variation in a particular $\mathcal{B}$-class, and we shall employ Theorem 2.2 for this purpose. First we prove the following auxiliary result:

<u>LEMMA 2.3</u> (Smoothing Magic) *Let* $F(x) \in \mathcal{D}(M;2)$ *for some* $M(x) \in \mathcal{M}$. *Then*

$$(2.2.8) \qquad \frac{[1 - F^{\ddagger}_{(1)}(\theta)]^2}{-i\theta} \in \mathcal{B}^{\ddagger}(M;\ 1).$$

<u>PROOF</u>. We want to show that

$$(2.2.9) \qquad \frac{2[1-- F^{\ddagger}_{(1)}(\theta)]}{-i\theta} - \frac{1 - [F^{\ddagger}_{(1)}(\theta)]^2}{-i\theta} \in \mathcal{B}^{\ddagger}(M;\ 1).$$

Write $U(x)$ for the unit function $P\{0 \le x\}$, and define

$$F^c_{(1)}(x) = U(x) - F_{(1)}(x)$$

$$F^c_{(1)2}(x) = U(x) - \int_{-\infty}^{\infty} F_{(1)}(x-z)dF_{(1)}(z).$$

Then proving (2.2.9) is equivalent to showing that

$$2F^c_{(1)}(x) - F^c_{(1)2}(x) \in L(M;\ 1). \tag{2.2.10}$$

The following respresentation for $F^c_{(1)2}(x)$ requires no proof, having an obvious interpretation:

$$F^c_{(1)2}(x) = \left\{F^c_{(1)}\left(\frac{x}{2}\right)\right\}^2 + 2\int_0^{x/2} F^c_{(1)}(x-z)\left[\frac{1-F(z)}{\mu_1}\right]dz\ .$$

Consequently showing (2.2.10) amounts to proving that

$$2F^c_{(1)}(x) - \left\{F^c_{(1)}\left(\frac{x}{2}\right)\right\}^2 - 2\int_0^{x/2} F^c_{(1)}(x-z)\left[\frac{1-F(z)}{\mu_1}\right]dz \in L(M;\ 1).$$

Since $\dfrac{1-F(x)}{\mu_1} \in L(M;\ 1)$ and

$$\int_0^x M(u)du \le xM(x),$$

we have

$$\int_0^{\infty}\int_0^x M(u)\ \frac{1-F(x)}{\mu_1}\ dudx < \infty.$$

By Fubini's theorem for nonnegative functions it follows that

$$\int_0^{\infty} M(x)F^c_{(1)}(x)dx < \infty.$$

On the other hand, since

$$\int_x^\infty uM(u)\left[\frac{1-F(u)}{\mu_1}\right]du \to 0$$

as $x \to \infty$, we have $xM(x)F^c_{(1)}(x) \to 0$. Therefore

$$\int_0^\infty x\{M(x)F^c_{(1)}(x)\}^2 dx < \infty.$$

Consequently by a change of variable

$$\int_0^\infty x\{M(\tfrac{x}{2})F^c_{(1)}(\tfrac{x}{2})\}^2 dx < \infty,$$

and since $M(x) \le M(\frac{x}{2})M(\frac{x}{2})$, it follows that

$$[F^c_{(1)}(\tfrac{x}{2})]^2 \in L(M; 1). \tag{2.2.11}$$

Since $F(x)$ has a finite variance,

$$\int_0^\infty F^c_{(1)}(\tfrac{u}{2})du < \infty,$$

and since $u\,M(u)F^c_{(1)}(u) \to 0$ as $u \to \infty$, we obtain

$$F^c_{(1)}(\tfrac{x}{2})F^c_{(1)}(x) \in L(M; 1). \tag{2.2.12}$$

Finally a second application of Fubini's theorem yields

$$\int_0^\infty xM(x)\left\{\int_0^{x/2}[F^c_{(1)}(x-z)-F^c_{(1)}(x)]\left[\frac{1-F(z)}{\mu_1}\right]dz\right\}dx$$

$$\le \int_0^\infty xM(x)\int_0^{x/2} z\left[\frac{1-F(x-z)}{\mu_1}\right]\left[\frac{1-F(z)}{\mu_1}\right]dzdx$$

$$= \int_0^\infty z \; \frac{1-F(z)}{\mu_1} \int_{2z}^\infty x \; M(x) \; \frac{1-F(x-z)}{\mu_1} \; dxdz$$

$$(2.2.13) \qquad = \int_0^\infty z \; \frac{1-F(z)}{\mu_1} \int_z^\infty (y+z)M(y+z) \; \frac{1-F(y)}{\mu_1} \; dydz.$$

Over the range of integration $y \geq z$ we have $M(y+z) \leq M(y)M(z)$. Consequently the integral in (2.2.13) is less than or equal to

$$2\int_0^\infty zM(z) \; \frac{1-F(z)}{\mu_1} \int_z^\infty yM(y) \; \frac{1-F(y)}{\mu_1} \; dydz$$

which is finite, since $\frac{1-F(x)}{\mu_1} \in L(M;\ 1)$. Consequently

$$(2.2.14) \qquad \int_0^{x/2} [F^c_{(1)}(x-z) - F^c_{(1)}(x)][\frac{1-F(z)}{\mu_1}] \; dz \in L(M;\ 1).$$

(2.2.11), (2.2.12) and (2.2.14) together imply (2.2.10), proving the lemma.

We note at this point that the result of Lemma 2.3 is quite remarkable. In view of the fact that $[1 - F^{\ddagger}_{(1)}(\theta)]^2$ is in the class $B^{\ddagger}(M,\ 1)$, we might reasonably conclude that $[1 - F^{\ddagger}_{(1)}(\theta)]^2/(-i\theta)$ is in the class $B^{\ddagger}(M;\ 0)$; i.e. we would expect to *lose* one whole moment as the price to be paid for dividing by $-i\theta$. The convolution in the numerator of (2.2.8) apparently causes the "lost" moment to "reappear", and we refer to this surprising efficacy as *smoothing magic*.

Now write $q^{\dagger}(\theta)$ for the special SMF $q^{\dagger}(\theta;\ -2,\ -1,\ 1,\ 2)$ and define

$$L_1^{\ddagger}(\theta) = q^{\dagger}(\theta)\, L^{\ddagger}(\theta)$$

and $$L_2^{\ddagger}(\theta) = [1 - q^{\dagger}(\theta)]L^{\ddagger}(\theta).$$

Clearly

$$L_1^{\ddagger}(\theta) = \frac{[1-F_{(1)}^{\ddagger}(\theta)]^2}{-\mu_1 i\theta} \frac{q^{\dagger}(\theta)}{F_{(1)}^{\ddagger}(\theta)} .$$

Set $J = [-2, 2]$ and $\Phi(z) = \frac{1}{z}$. Since $F_{(1)}^{\ddagger}(\theta)$ is continuous at the origin, $|F_{(1)}(\theta)|$ is bounded away from zero in a neighborhood of the origin say $(-\delta, \delta)$. On the other hand

$$\inf_{\substack{\delta\le\theta\le 2\\ -2\le\theta\le-\delta}} |1 - F^{\ddagger}(\theta)| = \lambda > 0,$$

since otherwise $F(x)$ would be a lattice distribution. Consequently

$$\inf_{\substack{\delta\le\theta\le 2\\ -2\le\theta\le-\delta}} \frac{|1 - F^{\ddagger}(\theta)|}{|-\mu_1 i\theta|} = \frac{\lambda}{\mu_1} > 0.$$

Thus for $\theta \in J$, $z = F_{(1)}^{\ddagger}(\theta)$ maps out a continuous curve that lies outside the circle

$$|z| = \frac{\lambda}{\mu_1} .$$

Using Part A of Theorem 2.2 we conclude that $q^{\dagger}(\theta)/F_{(1)}^{\ddagger}(\theta)$ is in $\mathcal{B}^{\ddagger}(M; 1)$. It then follows by the smoothing magic of Lemma 2.3 that $L_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 1)$.

It is somewhat easier to deal with

$$L_2^{\ddagger}(\theta) = [1 - q^{\dagger}(\theta)] \frac{[1 - F_{(1)}^{\ddagger}(\theta)]^2}{1 - F^{\ddagger}(\theta)} .$$

Here take $J = (-\infty, 2) \cup (2, \infty)$ and $\Phi(z) = \frac{1}{1-z}$. Since $F^{\ddagger}(\theta) \in C^{\ddagger}$, it is not difficult to see that $\rho[F^{\ddagger}(\theta)] < 1$. As θ runs through J,

$|F^{\dagger}(\theta)|$ is bounded away from unity. By Part B of Theorem 2.2

$$\frac{1 - q^{\dagger}(\theta)}{1 - F^{\dagger}(\theta)} \in \mathcal{B}^{\dagger}(M;\ 1),$$

and it follows that $L_2^{\dagger}(\theta) \in \mathcal{B}^{\dagger}(M;\ 1)$. Consequently

$$L^{\dagger}(\theta) = L_1^{\dagger}(\theta) + L_2^{\dagger}(\theta) \in \mathcal{B}^{\dagger}(M;\ 1).$$

We have shown that

$$\frac{1}{1-F^{\dagger}(\theta)} - \frac{1}{-\mu_1 i\theta}$$

is the Fourier-Stieltjes transform of $\frac{\mu_2}{2\mu_1^2} F_{(2)}(x) + L(x)$ where

$L(x) \in \mathcal{B}(M;\ 1)$. Consequently

$$\Delta_a^{\dagger}(\theta) \left\{\frac{1}{1-F^{\dagger}(\theta)} - \frac{1}{-\mu_1 i\theta} - 1\right\}$$

is the Fourier transform of

$$\int_{-\infty}^{\infty} \Delta_a(x-z)\ d\left\{\frac{\mu_2}{2\mu_1^2} F_{(2)}(z) + L(z)\right\} - \int_{-\infty}^{\infty} \Delta_a(x-z) dU(z).$$

From (2.2.6) it follows that

$$H(x) - \int_{-\infty}^{\infty} \Delta_a(x-z)\ d\left\{\frac{\mu_2}{2\mu_1^2} F_{(2)}(z) + L(z)\right\}$$

$$= \overline{H}(x) - \int_{-\infty}^{\infty} \Delta_a(x-z) dU(z)$$

But $\quad \overline{H}(x) = \int_0^{\infty} \Delta_a(x-z) \frac{dz}{\mu_1} + \Delta_a(x)$, so that

$$\int_{-\infty}^{\infty} \Delta_a(x-z)\ d\left\{\sum_{n=1}^{\infty} F_n(z)\right\}$$

$$= \int_{-\infty}^{\infty} \Delta_a(x-z)\left\{ U(z)\,\frac{dz}{\mu_1} + d\left[-\frac{\mu_2}{-2\mu_1^2}\; F_{(2)}(z) + L(z)\right] - dU(z)\right\} .$$

$\Delta_a(x)$ may be replaced by linear combinations of triangular densities, and these, in turn, may be used to approximate characteristic functions of intervals by a "sandwiching" process described by Smith (1964). By a standard extension argument we then obtain

$$H(x) = \frac{x}{\mu_1}\, U(x) + \frac{\mu_2}{2\mu_1^2}\; F_{(2)}(x) + L(x) - U(x) + C.$$

Since $\mu_1 > 0$ it follows from the Strong Law of Large Numbers that $H(x) \to 0$ as $x \to -\infty$. Therefore $C = -L(-\infty) = 0$.

We now state the renewal theorem which has been the aim of the preceding discussion:

THEOREM 2.4 *Suppose that* $\{X_n\}_{n=1}^{\infty}$ *is a sequence of iid positive random variables with distribution function* $F(x) \in \mathcal{D}(M;\, 2)_\cap C$ *for some* $M(x) \in M$. *Then*

$$H(x) = \sum_{n=1}^{\infty} P\{X_1 + \ldots + X_n \le x\} = \left(\frac{x}{\mu_1} - 1\right)U(x) + \frac{\mu_2}{2\mu_1^2}\, F_{(2)}(x) + L(x),$$

where $L(x)$ *is a function of bounded variation in the class* $B(M;\, 1)$, $L(-\infty) = 0$, *and, in particular*,

$$L(x) = o\left(\frac{1}{xM(x)}\right)$$

as $x \to \infty$.

Reference to renewal theorems of this type (involving a derived distribution term in the expansion) appears to have been made first by Smith (1967); see the addendum to that paper. The version stated above seems to be the most general result *with respect to the moment class* M that can be obtained via the Wiener-Pitt-Lévy-Smith approach. The proof of Theorem 2.4 does suggest extensions in other directions which will be taken up in subsequent chapters. In Chapter 4 we shall discover some smoothing magic in an investigation of the renewal function, assuming $F(x) \in \mathcal{D}(M; 1 + \alpha)$, $0 < \alpha < 1$. This will involve adding more terms to the expansion in (2.2.7). In Chapter 5 we shall examine cumulants of the renewal process (and, in particular, the variance,) when more than a second moment is assumed for $F(x)$.

CHAPTER III: AN APPLICATION TO SUPERPOSITION OF RENEWAL PROCESSES

Having examined a fundamental problem in renewal theory in the previous chapter, we shall now apply the main result (Theorem 2.4) of that discussion in a brief study of superposed renewal processes. The literature concerning superposition is quite extensive, and consequently we have confined the review in Section 3.1 to a small number of references. Section 3.2 deals with the probabilistic behavior of superposition under transient conditions, an aspect which has largely been ignored by previous authors.

3.1 A Selective Review

Suppose that N independent sources each give rise to a series of events and that the outputs of these sources are superimposed into a single pooled output. The superposition process is thus a series of events in which an event occurs at time t iff an event occurs at t in at least one of the N component processes. This model was first studied by Cox and Smith (1953) in connection with a problem in neuro-physiology; however, it has also arisen in a number of other diverse areas of application, including the theory of congestion in telephone traffic, investigations of computer failure patterns, and studies of multi-stage industrial processes involving similar machines operating in parallel.

Cox and Smith (1953) dealt with the superposition of strictly periodic sequences of events, i.e., when the events from the ith source occur

exactly at times θ_i, $2\theta_i$, $3\theta_i$, ..., where θ_i is the period of the ith source, $(i = 1,2,\ldots,N,)$ and the periods are mutually irrational. Although this is a deterministic situation, it can be shown that asymptotically as N becomes large, the pooled output (viewed over a long period of time) becomes indistinguishable from a Poisson process with parameter $\lambda = \sum_{i=1}^{N} (1/\theta_i)$. Mild assumptions are made to ensure that as N tends to infinity, the time scale is dilated so that no small group of periods θ_i is comparable with the mean interval between events in the superposition.

A consequence of this rather surprising result is that *for large N* the analysis of local behavior of the superposition yields little information regarding the individual sources. It is well known (and easy to demonstrate) that the superposition of p independent Poisson processes each of parameter λ is itself a Poisson process of parameter $p\lambda$. Therefore no firm conclusions can be drawn from the analysis of a pooled output which does not differ significantly from a Poisson process.

For small N it is possible, at least in principle, to estimate the θ_i's accurately, provided the pooled series available for analysis is long. The procedure consists of decomposing the frequency distribution of the intervals between successive events. For larger values of N, Cox and Smith (1953) proposed an analysis based on the variance-time curve $V_N(t) = \mathrm{Var}\{$number of events occurring in the superposition during $[0,t]\}$. $V_N(t)$, which can be estimated from observations on the superposition, oscillates about $N/6$ for large t iff the series is the pooled outlet of N periodic sources.

More generally we wish to study the behavior of the superposition when the sources form independent renewal processes. The superposition

will not, in general, be a renewal process, since the intervals between events are not necessarily independent random variables. (In fact, if the superposition of two independent renewal processes with the same interoccurrence distribution $F(x)$ having mean $\mu < \infty$ is also a renewal process, then all three processes are Poisson; see, e.g., Karlin and Taylor (1975, page 226).)

Cox and Smith (1954) assumed that for each source the intervals between successive events form a sequence of iid positive random variables with distribution $F(x)$. (We shall refer to this as the identically distributed case.) They considered the equilibrium behavior of the superposition a long time after the start of the process as follows:

For $i = 1,2,\ldots,N$ let Y_i be the time measured back from a fixed sampling point to the preceding event on the ith source. Define Y as the corresponding random variable for the *superposition*. Then

$$Y = \min(Y_1,\ldots,Y_N).$$

For renewal processes in equilibrium the Key Renewal Theorem can be used to show that

$$P\{Y_i > y\} = \int_y^\infty \frac{1-F(x)}{\mu}\,dx = 1 - F_{(1)}(y), \tag{3.1.1}$$

provided that $\mu = \int_0^\infty x\,dF(x) < \infty$. (Note that (3.1.1) does not depend on the choice of the sampling point.) Then by independence

$$P\{Y > y\} = [1 - F_{(1)}(y)]^N .$$

The density corresponding to the backward delay distribution for the superposition is given by

$$\frac{N}{\mu}\,[1 - F(y)]\,[1 - F_{(1)}(y)]^{N-1} \quad . \tag{3.1.2}$$

Now let $G(x)$ denote the (equilibrium) distribution of the interval between events in the superposition, and let $g(x)$ be the corresponding density. Clearly the mean interval between events in the superposition is μ/N. Suppose we take a sampling point chosen at random over a very long time interval and define Z as the time measured back to the last event in the superposition. If X denotes the length of the inter-event interval in which the sampling point lies, then X has the length-biased density $xg(x)/(\mu/N)$. The conditional density of Z given that $X = x_o$ is uniform over $(0, x_o)$. Thus the unconditional pdf of Z is

$$\int_y^\infty \frac{1}{x_o} \cdot \frac{x_o g(x_o)}{\mu/N}\, dx_o = \frac{1-G(y)}{\mu/N} \quad . \tag{3.1.3}$$

Because the renewal processes are in equilibrium, the expressions (3.1.2) and (3.1.3) are equivalent, and consequently

$$1 - G(y) = [1 - F(y)]\,[1 - F_{(1)}(y)]^{N-1}. \tag{3.1.4}$$

Write L for the length of an interval between consecutive events in the pooled output under equilibrium. (Note that dividing L by $E(L) = \mu/N$ corresponds to a dilation of the time scale.) Then assuming $F(0) = 0$ (3.1.4) implies that as $N \to \infty$,

$$P\left\{\frac{L}{\mu/N} > y\right\} = [1 - F(\tfrac{y\mu}{N})]\,[1 - F_{(1)}(\tfrac{y\mu}{N})]^{N-1}$$

$$= [1 - F(\tfrac{y\mu}{N})] \left\{1 - \int_0^{y\mu/N} \frac{1-F(u)}{\mu}\, du\right\}^{N-1}$$

$$\to e^{-y} \; .$$

Cox and Smith (1954) showed equivalently that as N tends to infinity the limit distribution of the number of events occurring in an interval of length $\mu\tau/N$ is Poisson with parameter τ. They made the mild assumption that there exists β, $0 < \beta < 1$, such that $F(t) = O(t^{\beta})$ for small t. It can be shown more generally that in the limit the numbers of events in non-overlapping intervals are independent, i.e., the superposition is a Poisson process. Such results provide a theoretical basis for making a "Poisson assumption" in a number of applications, just as the Central Limit Theorem is frequently invoked to justify assumptions of normality.

Cox and Smith (1954) pointed out the difficulty of analyzing a superposition when N is large, noting that the sources might equally well be strictly periodic or renewal processes. They suggested a variance-time curve analysis for situations in which N is small and unknown. (cf. Cox and Lewis (1966, page 215).) Let μ_1, σ, and μ_3 refer to $F(x)$. Then

$$V_N(t) - \frac{N\sigma^2 t}{\mu_1^3} \sim N\left(\frac{1}{6} + \frac{\sigma^4}{2\mu_1^4} - \frac{\mu_3}{3\mu_1^3}\right)$$

for very large t. Comparison with the empirical variance-time curve yields estimates for the four unknown parameters. However these are based on sample estimates of higher moments which tend to be unreliable. Cox and Lewis (1966) additionally noted that this approach is likely to be inefficient because transient effects are not taken into account, suggesting an area for further investigation. We shall return to several related issues later in Sections 5.3, 5.4, and 6.2.

Since 1954 very few authors have dealt with the inferential aspect of superposition. On the other hand, limit theorems have received

increasing attention, and most recently there has been a surge of interest in the superposition of very general types of point processes.

Khintchine (1960, Chapter 5) observed that the process describing the arrival of calls in a telephone exchange often approximates a Poisson process more closely than might be expected. He explained this phenomenon by showing that the superposition of indefinitely many uniformly sparse but not necessarily identical renewal processes tends to a Poisson process. Since the statement of this result in the English translation of Khintchine's monograph is somewhat awkward, we quote the following recent version due to Karlin and Taylor (1975, Chapter 5):

For $n = 1,2,\ldots,$ and for $i = 1,\ldots,k_n$, where $k_n \to \infty$ as $n \to \infty$, let $N_{ni}(t)$ be a renewal counting process with underlying distribution $F_{ni}(t)$. (For every n the processes $\{N_{n1}(t)\},\ldots,\{N_{nk_n}(t)\}$ are assumed to be independent.) The superposition process $N_n(t)$ is defined as

$$N_n(t) = \sum_{i=1}^{k_n} N_{ni}(t), \quad t \ge 0.$$

Then according to Khintchine's theorem,

$$\lim_{n\to\infty} P\{N_n(t)=j\} = \frac{e^{-\lambda t}(\lambda t)^j}{j!}, \quad (j = 0,1,\ldots)$$

if and only if

$$\lim_{n\to\infty} \sum_{i=1}^{k_n} F_{ni}(t) = \lambda t,$$

provided that $\lim\limits_{n\to\infty} \max\limits_{1\le i\le k_n} F_{ni}(t) = 0.$

Franken (1963) proved a refinement of this result for the iid case.

Let

$$\xi_{ni} = N_{ni}(u_o+t) - N_{ni}(u_o)$$

for $i = 1,\ldots,n$ and $n = 1,2,\ldots,$ where the underlying distributions $F_{ni}(t)$ are for $i = 1,\ldots,n$ identically equal to $F_n(t)$. Define

$$\zeta_n = \sum_{i=1}^{n} \xi_{ni} ,$$

and assume that $nH_n(t) = H(t)$ for all n and t, where $H_n(t)$ is the renewal function corresponding to $F_n(t)$, and $H(t)$ is an arbitrary renewal function. Franken (1963) showed that if $F_n(+0) < 1/2$, then

$$\sum_{0\le k<x} P\{\zeta_n=k\} = \sum_{o\le k<x} \psi(k)\left[1 + \sum_{i=1}^{r} \frac{Q_i(k)}{n^i}\right] + 0\left[\frac{1}{n^{r+1}}\right] ,$$

where the $Q_i(k)$ are certain calculable polynomials in k, $r = 0,1,\ldots$. If $F_n(+0) < 1/3$, then the estimates so obtained are uniform in x.

Franken's proof (which is quite complicated) is based on a generating function approach which makes use of a Charlier Type B series. Unfortunately the condition "$nH_n(t) = H(t)$" is unrealistic, ruling out use of the expansion in many applications. This restriction has apparently been overlooked by authors quoting Franken's result; see, for example, the review by Cinlar (1972).

Ambartzumian (1965) discussed two "inverse problems" related to the superposition of renewal processes: (1) determining N and $F(x)$ when the sources consist of N identical renewal processes, and (2) determining $F(x)$ and λ when the component processes consist of a Poisson process with parameter λ and a renewal process with underlying distribution $F(x)$. Ambartzumian's results are stated vaguely, and no numerical examples are

presented. His "general method" for the first problem does not go beyond the work of Cox and Smith (1954), nor is it clear that his "method of moments" answers the second question satisfactorily in a statistical sense.

In a second paper Ambartzumian (1969) discussed the correlation properties of the intervals in the superposition of N not necessarily identical renewal processes in equilibrium. Let $X_o, X_1, X_2, \ldots$ denote the lengths of consecutive intervals between events in the superposition, and let

$$\phi(z) = \sum_{n=1}^{\infty} \rho_n z^n ,$$

where $\rho_k = \mathrm{Corr}(X_o, X_k)$. Also let $F_i(x)$ denote the underlying distribution for the ith renewal process. Assume that (A) for $i = 2,3,\ldots,N$ the Laplace transforms

$$\ell_i(s) = \int_o^{\infty} e^{-sx}\, dF_i(x)$$

converge in the strip $-a < \mathrm{Re}s < 0$ for some $a > 0$, and (B) each $F_i(x)$ possesses an absolutely continuous component. Then (in the somewhat confusing notation of Ambartzumian)

$$\frac{c_o}{2} + (c_o - 1)\phi(1) = \sum_{K=1}^{N} \frac{\mu_k \sigma_k^3}{2} ,$$

where $c_o = EX_o^2$, $\mu_k = \int_o^{\infty} x^2\, dF_k(x)$, and $\sigma_k^{-1} = \int_o^{\infty} x dF_k(x)$. Ambartzumian did not, however, extract closed expressions for the serial correlations.

Blumenthal, Greenwood, and Herbach (1968) considered the rate of convergence to the exponential form of the inter-event distribution for the superposition, both as a function of time and as a function of N,

the number of component processes. They examined both the iid and the non-identical cases, assuming that the underlying distributions are members of the gamma family with selected shape parameters. By making such concrete assumptions, Blumenthal, Greenwood, and Herbach were able to generate a number of interesting plots which enabled them to rate the effects of factors such as system age, system size and shape of the distribution on deviation from the exponential limit. The most important conclusion was that system age can cause very large deviations from the limit, and that these can extend over a relatively long time span. The work of Blumenthal, Greenwood, and Herbach is apparently the first to deal with the transient aspects of superposition from a practical standpoint.

Lawrance (1973) studied the dependence of intervals between events in the superposition of independent (not necessarily identical) stationary point processes. In particular he obtained the joint distribution of any number of adjacent inter-event intervals following an arbitrary event in the superposition. Although this distribution can be stated in a closed form, the result is notationally intractable. In fact Lawrance found it necessary to restrict his study to the joint distributions and serial correlations of at most three adjacent intervals. His work includes some interesting numerical results based on superpositions of Erlang renewal processes.

Recently Coleman (1976) dealt with a special type of superposition involving dependence. Suppose we start with a renewal process in equilibrium and choose independently a sampling point. The distances from this point measured forward and backward to renewal points define two new renewal processes. Their superposition constitutes a folding over of the past of the original process onto its future, and these are independent

only in the Poisson case. Coleman obtained the joint distribution of k adjacent intervals for this rather unusual example of superposition.

3.2 The Transient Behavior of Superposition: A Renewal Theoretic Approach

The studies reviewed above have, for the most part, dealt with probabilistic aspects of superposition of renewal processes. We believe that one goal of such work should be to provide insights which lead to improved statistical methodology, and there is clearly a lack of results which are useful in this sense. Inference about the source processes based on observation of the pooled output is feasible only when the superposition differs significantly from a Poisson process. This suggests the need for further investigation of superposition of a relatively small number of processes under time dependent (rather than equilibrium) conditions.

Although we shall not attempt to develop statistical techniques for dealing with data arising from superposition, Theorem 2.4 can be used to derive certain results for the transient case which may well lead to more precise statistical work in the future. Specifically we shall consider the relatively simple problem of finding the probability that k events occur in an aribitrary interval of time for a superposition of N independent, identical renewal processes $\{X_i\}_{i=1}^{\infty}$. (Apparently this issue has been overlooked by previous authors, including Blumenthal, Greenwood, and Harbach (1968).)

Let $F(x) = P\{X_i \le x\}$, and write $H(x)$ for the corresponding renewal function. We shall assume that

(1) N, the number of sources, is fixed and relatively small, (so that it cannot be regarded as tending to infinity),

(2) the first renewal lifetime X_1 for each component process begins at time $t = 0$,

and (3) by time $t = t_o$ the component renewal processes have not yet reached equilibrium.

Furthermore we shall assume that $F(x) \in \mathcal{D}(M; 2)_{\cap} C$ for some moment function $M(x) \in \mathcal{M}$, using μ_1 and μ_2 to denote the first two moments of $F(x)$.

Let $\Delta t > 0$ be a fixed increment of time, and for $j = 0,1,2,\ldots,$ and $k = 0,1,2,\ldots,$ define

$$p_j = P\{\text{exactly } j \text{ renewals occur in } (t_o, t_o+\Delta t] \text{ for given component process}\}$$

and

$$P_k = P\{\text{exactly } k \text{ renewals occur in } (t_o, t_o+\Delta t] \text{ for the superposition}\}.$$

Studies of the limiting behavior of superposition as N becomes large typically involve a standardization which can be thought of as a dilation of the time scale for the sources. Specifically the process $\{X_n\}_{n=1}^{\infty}$ is replaced by $\{NX_n/\mu_1\}_{n=1}^{\infty}$. We shall follow this procedure here for the sake of comparison with the limiting Poisson distribution which we shall write as $\{P_k^*\}_{k=0}^{\infty}$.

The probability that no event occurs in $(t_o, t_o+\Delta t]$ for the process $\{NX_n/\mu_1\}_{n=1}^{\infty}$ is

$$p_o = \left[1 - F\left(\frac{t_o+\Delta t}{N/\mu_1}\right)\right] + \int_o^{\mu_1 t_o/N} \left[1 - F\left(\frac{t_o+\Delta t}{N/\mu_1} - u\right)\right] dH(u). \tag{3.2.1}$$

Note that the first term on the right-hand side of (3.2.1) is the probability that the first renewal takes place after time $t_o + \Delta t$; the second term is the probability that a renewal occurs in $(0, t_o]$ such that the next renewal takes place after time $t_o + \Delta t$.

For convenience write

$$t_o' = \mu_1 t_o/N \quad \text{and} \quad \delta = \mu_1(\Delta t)/N .$$

Using the independence of the sources

$$P_o = \left\{1 - F(t_o' + \delta) + H(t_o') - \int_o^{t_o'+\delta} F(t_o' + \delta - u)\,dH(u) + \int_{t_o'}^{t_o'+\delta} F(t_o' + \delta - u)\,dH(u)\right\}^N . \tag{3.2.2}$$

(3.2.2) can be simplified by applying the integral equation of renewal theory; consequently

$$P_o = \left\{1 - \int_{t_o'}^{t_o'+\delta} [1 - F(t_o' + \delta - u]\,dH(u)\right\}^N . \tag{3.2.3}$$

(3.2.3) is a particularly revealing expression which does not appear to have been studied elsewhere. It involves both the behavior of $F(x)$ near the origin and the behavior of $H(x)$ for moderately large values of x.

We shall deal with the former by assuming that there exist positive constants λ and ρ such that

$$F(x) = \lambda x^\rho + 0(x^{\rho+1}) \quad \text{as } x \to 0. \tag{3.2.4}$$

This is a fairly mild condition, in view of the fact that (3.2.4) is satisfied by a number of useful lifetime distributions, including the uniform, negative exponential, gamma, and Weibull families.

To handle the transient behavior of $H(x)$ we apply Theorem 2.4 of the previous chapter. Substitution into (3.2.3) yields

$$P_o = \left\{1 - \int_{t_o'}^{t_o'+\delta} [1 - F(t_o'+\delta-u)]\left[\frac{du}{\mu_1} + \frac{\mu_2}{2\mu_1^2} dF_{(2)}(u) + dL(u)\right]\right\}^N, \tag{3.2.5}$$

where the remainder function $L(x)$ belongs to $\mathcal{B}(M;\ 1)$. Note that the bounded variation property of $L(x)$ is essential to this application.

Now by assumption (3.2.4),

$$\frac{1}{\mu_1}\int_{t_o'}^{t_o'+\delta} [1 - F(t_o'+\delta-u)]du = \frac{\delta}{\mu_1} - \frac{\lambda\delta^{\rho+1}}{\mu_1(\rho+1)} + 0(\delta^{\rho+2}) .$$

Since the second derived density is given by

$$f_{(2)}(u) = \frac{2}{\mu_2}\int_u^{\infty} (v-u)dF(v),$$

it follows that

$$\begin{aligned}\frac{\mu_2}{2\mu_1^2}\int_{t_o'}^{t_o'+\delta} [1 - F(t_o'+\delta-u)]dF_{(2)}(u) &= \\ &= \frac{1}{\mu_1^2}\int_{t_o'}^{t_o'+\delta}\int_u^{\infty} [1 - F(t_o'+\delta-u)](v-u)dF(v)du.\end{aligned} \tag{3.2.6}$$

After interchanging the order of integration, simplifying and collecting terms, (3.2.6) reduces to

$$\frac{1}{2\mu_1^2}\int_0^\delta u^2 dF(u+t_o') + \frac{\delta}{\mu_1^2}\int_\delta^\infty u dF(u+t_o')$$

$$- \frac{\delta^2}{2\mu_1^2}[1 - F(t_o'+\delta)] - \frac{\lambda\delta^{\rho+1}}{\mu_1^2(\rho+1)}\int_\delta^\infty u dF(u+t_o') + O(\delta^{\rho+2}) .$$

The most intractable term in (3.2.5) is

(3.2.7)
$$\int_{t_o'}^{t_o'+\delta}[1 - F(t_o'+\delta-u)]dL(u) = L(t_o'+\delta) - L(t_o')$$
$$- \lambda\int_0^\delta (\delta-v)^\rho dL(v+t_o') + O\left(\int_0^\delta (\delta-v)^{\rho+1} dL(v+t_o')\right) .$$

The difference $L(t_o'+\delta) - L(t_o')$ is not amenable to a Taylor expansion unless $F(x)$ possesses additional smoothness. We prefer instead to assume that

(3.2.8)
$$\frac{1}{t_o'^2 M(t_o')} = O(\delta^{\rho+2}),$$

which should not be unnecessarily restrictive, especially in applications where $F(x)$ may well have more than a second moment.

We shall use (3.2.8) in connection with the following lemma which implies that

$$L(t_o'+\delta) - L(t_o') = o\left(\frac{1}{t_o'^2 M(t_o')}\right),$$

provided that $M(x)$ is, additionally, a special type of function of moderate growth.

LEMMA 3.1 *Let* $M(x) \in \mathcal{M}$ *and suppose that*

$$\frac{M(x+c)}{M(x)} = 1 + O\left(\frac{1}{x}\right) \quad \text{as} \quad x \to \infty \tag{3.2.9}$$

for every constant $c > 0$. *Then*

$$L(x+\delta) - L(x) \in \mathcal{B}(M;\ 2). \tag{3.2.10}$$

PROOF. By a change of variable,

$$\begin{aligned} \int_0^\infty x^2 M(x)\, d\{L(x+\delta) - L(x)\} &= \int_\delta^\infty \{u^2 M(u-\delta) - u^2 M(u)\}\, dL(u) \\ &- 2\delta \int_\delta^\infty uM(u-\delta)\, dL(u) + \delta^2 \int_\delta^\infty M(u-\delta)\, dL(u)\ . \end{aligned} \tag{3.2.11}$$

Using the facts that $L(x) \in \mathcal{B}(M;\ 1)$ (by Theorem 2.4), $M(x)$ is a function of moderate growth, and

$$u^2 M(u-\delta) - u^2 M(u) = uM(u)\left\{u\left(\frac{M(u-\delta)}{M(u)} - 1\right)\right\},$$

we can apply dominated convergence to conclude that the integrals on the right-hand side of (3.2.11) are finite. Thus $L(x+\delta) - L(x) \in \mathcal{B}(M;\ 2)$ and, in particular, as $x \to \infty$,

$$|x^2 M(x)\,[L(x+\delta) - L(x)]| \le \int_x^\infty u^2 M(u)\,|d[L(u+\delta) - L(u)]| \to 0.$$

N.B. Various authors have obtained estimates for the function $H(x+1) - H(x)$, which is sometimes referred to as the "Blackwell difference"; see, for example, Theorem 3 of Stone (1965). We can modify Theorem 2.4 in an obvious manner to obtain an expansion for the Blackwell difference, and the remainder term $L(x+1) - L(x)$ then belongs to $\mathcal{B}(M; 2)$ according to Lemma 3.1, provided, of course, that $F(x) \in \mathcal{D}(M, 2)$ and $M(x)$ satisfies (3.2.9). Lemma 3.1 thus improves the result of Stone (1965) who showed that

$$L(x+1) - L(x) = o\left(\frac{x^2}{\log x}\right) \quad \text{as} \quad x \to \infty$$

if $F(x) \in \mathcal{D}(I; 2)$. We suspect, moreover, that (3.2.10) is true for all $M(x) \in M$.

Returning to (3.2.7), assumptions (3.2.8) and (3.2.9) imply that

$$\int_{t_o'}^{t_o'+\delta} [1 - F(t_o'+\delta-u)]dL(u) = O(\delta^{\rho+2}).$$

By expanding the right-hand side of (3.2.3) and collecting terms of the same order of magnitude we obtain the following estimate:

For fixed N *and* t_o, *as* $\Delta t \to 0$,

$$(3.2.12) \quad P_0 = 1 - (\Delta t)\left[1 + \frac{1}{\mu_1}\int_\delta^\infty u\,dF(u + t_o')\right]$$

$$+ \frac{(\Delta t)^{1+\rho}}{N^\rho}\left(\frac{\lambda\mu_1^\rho}{1+\rho}\right)\left[1 + \frac{1}{\mu_1}\int_\delta^\infty u\,dF(u+t_o')\right]$$

$$+ \frac{(\Delta t)^2}{2N}[1 - F(t_o'+\delta)] - \frac{N}{2\mu_1^2}\int_o^\delta u^2 dF(u+t_o')$$

$$+ (\Delta t)^2 \cdot \frac{N-2}{2N}\left[1 + \frac{1}{\mu_1}\int_\delta^\infty u dF(u+t_o')\right]^2 + 0(\Delta t)^{\rho+2} .$$

The approximation (3.2.12) may be too complicated for practical purposes; however (3.2.12) does, in fact, simplify to

$$P_o = 1 - (\Delta t)\left[1 + \frac{1}{\mu_1}\int_\delta^\infty u dF(u+t_o')\right] + 0\left[(\Delta t)^{1+\rho}\right],$$

so that by comparison with the limiting Poisson case,

$$P_o = P_o^* - \frac{\Delta t}{\mu_1}\int_\delta^\infty u dF(u+t_o') + 0\left[(\Delta t)^{1+\rho}\right] . \tag{3.2.13}$$

Expression (3.2.13) is particularly interesting, since it reveals that N and t_o (rather than the behavior of $F(x)$ at the origin) determine the rate of convergence to the Poisson limit. Furthermore, the "correction term" in (3.2.13) is specifically related to the first moment of the distribution $F(x)$.

The approach outlined in this section can be used to derive approximations such as (3.2.12) for P_1, P_2, P_3, and so forth. The algebraic details become increasingly complicated, and for convenience we simply note that

$$P_1 = P_1^* + \frac{\Delta t}{\mu_1}\int_\delta^\infty u dF(u+t_o') + 0\left[(\Delta t)^{1+\rho}\right]$$

and for $k \geq 2$,

$$P_k = 0\left[(\Delta t)^{1+\rho}\right] .$$

These estimates demonstrate that the "smoothing magic" of Section 2.2 can lead to detailed results concerning superposition. We shall resume our discussion of this application in Section 5.3.

CHAPTER IV: A REPRESENTATION FOR THE RENEWAL FUNCTION WHEN THE VARIANCE IS INFINITE

In this chapter we resume our study of the asymptotic behavior of the renewal function $H(x)$. We shall develop an expansion for $H(x)$ assuming that $F(x)$, the underlying distribution, has an infinite variance and, furthermore, that $F(x) \in \mathcal{D}(M; 1+\delta) \cap C$, where $M(x) \in M^*$ and $0 < \delta < 1$. Some previous work related to this problem is reviewed in Section 4.1. The methods used to prove Theorem 2.4 are relevant to the present situation, although, as we show in Section 4.2, a different kind of "smoothing magic" is necessary. Our main result (Theorem 4.6) is contained in Section 4.3.

4.1 A Theorem of Stone and Wainger

In a study of the theory of recurrent events Stone and Wainger (1967) dealt with a lattice distribution $\{f_n\}_{n=-\infty}^{\infty}$ defined on the integers and with the corresponding renewal measure $\{u_n\}_{n=-\infty}^{\infty}$. (In the more familiar situation where $f_n = 0$ for $n < 0$, f_j is interpreted as the probability that an aperiodic event takes place for the first time at time j, and u_k is interpreted as the probability that a recurrent event occurs at time k.) Note that $\{f_n\}_{n=-\infty}^{\infty}$ and $\{u_n\}_{n=-\infty}^{\infty}$ are the analogues of the probability density and renewal intensity functions which arise when $F(x)$ is assumed to be absolutely continuous. Specifically Stone and Wainger considered the problem of estimating u_n, assuming that $\{f_n\}_{n=-\infty}^{\infty}$ has at least a finite nonzero first moment μ_1.

A brief description of their work is appropriate here, since the theories of recurrent events and renewals are parallel to a great extent.

Stone and Wainger introduced a class of moment functions $M(x)$ defined by the following properties:

(i) $2 \le M(x) < \infty$ for $0 \le x < \infty$,

(ii) $M(x)$ is nondecreasing,

(iii) $y^{-1} \log M(y) \le x^{-1} \log M(x)$ for $0 \le x < y < \infty$,

(iv) $x^{-1} \log M(x) \to 0$ as $x \to \infty$.

We shall refer to this class as M'. For part of their work Stone and Wainger required additional restrictions, including the condition $M(2x) = O(M(x))$ for all $x \ge 0$. M' is homologous to the classes M and M^* used in Chapter 2, although the latter seem to be more natural; conditions (iii) and (iv) are not obvious and were needed by Stone and Wainger for purely technical reasons. M' does include functions such as

$$x^{\alpha}(\log x)^{\beta}\, e^{x^{\delta}},$$

where $0 < \delta < 1$, and either $\gamma > 0$ or $\gamma = 0$ and $\alpha > 0$.

Two additional sequences are defined as follows: Let

$$q_o = \mu_1 - \sum_{j=1}^{\infty} f_j$$

and
$$q_k = -\sum_{j=k+1}^{\infty} f_j \quad \text{for } k > 0.$$

Also set

$$r_k = -\sum_{j=k+1}^{\infty} q_j \quad \text{for } k \ge 0.$$

Since μ_1 is finite, the sequence $\{q_k\}_{k=0}^{\infty}$ is summable. Note that if we assume, in addition, the existence of the second moment of $\{f_n\}_{n=-\infty}^{\infty}$,

then when $f_n = 0$ for $n < 0$, the sequences $\{q_k\}_{k=0}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ correspond to the first and second derived distributions, respectively. Write $q^{(m)}$ for the m-fold convolution of $\{q_k\}_{k=0}^{\infty}$ with itself.

For the one-sided case where $\{f_j\}_{j=-\infty}^{\infty}$ is concentrated on the non-negative integers, Stone and Wainger obtained the following result:

<u>THEOREM 4.1</u> (Stone and Wainger, 1967) *Assume that*

$$\sum_{j=0}^{\infty} j^{1+\delta} M(j)\, f_j < \infty$$

for some δ, $0 < \delta < 1$, *and some* $M(x) \in M'$. *Then for* n *sufficiently large, as* $k \to \infty$,

$$u_k - \frac{1}{\mu_1} = \sum_{m=1}^{n} \frac{(r * q^{(m-1)})_k}{\mu_1^{m+1}} + O(\rho(k)), \tag{4.1.1}$$

where for $k > 0$,

$$\rho(k) = \frac{1}{k^{2+2\delta} M(k)} + \frac{1}{kM(k)} \sum_{j=k/2}^{2k} \frac{jM(j) f_j}{1+|k-j|^{1+\delta}} .$$

In fact,

$$\rho(k) = o\left(\frac{1}{k^{1+\delta}\, M(k)}\right) \quad \text{as } k \to \infty. \tag{4.1.2}$$

Based on such minimal conditions the estimate (4.1.2) for the remainder term in the expansion is surprisingly sharp. Unfortunately the meaning of Theorem 4.1 is not entirely clear; the statement begs the question of the precise number n of terms to be included in the expansion

for a particular choice of δ, and this issue is not resolved in the rather sketchy proof given by Stone and Wainger.

Results similar to Theorem 4.1 were later obtained by Essén (1973), (although he, too, failed to specify n). Whereas Stone and Wainger used *ad hoc* techniques to obtain (4.1.1), Essén showed that a relatively simple proof is possible if certain commutative Banach algebras are first introduced. In particular, Essén derived approximations to u_k in which the remainder term is shown to be of "o" or "O" type, depending on corresponding assumptions made concerning the distribution $\{f_j\}_{j=-\infty}^{\infty}$. The Banach algebra method also yields results in renewal theory, but in this context it appears that no significant advantage is to be gained over the approach discussed in Chapter 2; (cf. the remark by Smith (1976; page 16)).

Theorem 4.1 does suggest an expansion representation for the renewal function $H(x)$, although we do not believe (as conjectured by Stone and Wainger) that the techniques used to obtain (4.1.1) can be readily adapted to the non-lattice case. Of course we have already established such a result for $H(x)$ when $F(x)$ has a finite second moment (see Theorem 2.4), and clearly the expansion in that case consists of a single term. Therefore we now consider the consequence of assuming that $F(x) \in \mathcal{D}(M; 1+\delta) \cap C$ for $0 < \delta < 1$ and some appropriate moment function $M(x)$, *and assuming, further more, that* $F(x)$ *has an infinite second moment.*

This problem was, in fact, the subject of a Ph.D. dissertation written by Dubman (1970) under the direction of Stone at the University of California at Los Angeles. Dubman made rather specialized assumptions concerning $F(x)$, such as

$$1 - F(x) \sim Cx^{-\alpha} \quad \text{as} \quad x \to \infty,$$

where C is a constant and $1 < \alpha < 2$, and,

$$F(x+1) - F(x) = 0(x^{-1-\alpha}) \quad \text{as} \quad x \to \infty.$$

Using Fourier analytic methods he obtained results of the form

$$H(x) = \sum_{m=1}^{n} \frac{1}{\mu_1^m} T_m(x) + o(1) \quad \text{as} \quad x \to \infty,$$

where for $x \geq 0$, $T_1(x) = x$ and

$$T_m(x) = \mu_1 T_{m-1}(x) - T_m(x) * [1-F(x)].$$

The number n of terms is specified to be such that $n \geq 2$ and $(n+1)/n < \alpha \leq n(n-1)$. Dubman did not allow for more general moment conditions. It is interesting to note that his techniques resemble methods used by Stone (1965) rather than the approach followed later by Stone and Wainger (1967).

4.2 A Preliminary Look at the Non-Lattice Case

Let $n \geq 1$ be some positive integer and consider the following formal expansion:

$$\frac{1}{1-F^{\ddagger}(\theta)} = \frac{1}{-\mu_1 i\theta F^{\ddagger}_{(1)}(\theta)} = \frac{1}{-\mu_1 i\theta}\left\{\frac{1}{1-[1-F^{\ddagger}_{(1)}(\theta)]}\right\}$$

$$= \frac{1}{-\mu_1 i\theta}\left\{1 + \sum_{j=1}^{n} [1-F^{\ddagger}_{(1)}(\theta)]^j + \frac{[1-F^{\ddagger}_{(1)}(\theta)]^{n+1}}{1-[1-F^{\ddagger}_{(1)}(\theta)]}\right\}$$

$$(4.2.1) \qquad = \frac{1}{-\mu_1 i\theta} + \sum_{j=1}^{n} \frac{[1-F^{\ddagger}_{(1)}(\theta)]^j}{-\mu_1 i\theta} + L^{\ddagger}_n(\theta),$$

say, where

$$(4.2.2) \qquad L^{\ddagger}_n(\theta) = \frac{[1-F^{\ddagger}_{(1)}(\theta)]^{n+1}}{1 - F^{\ddagger}(\theta)} .$$

This expansion is simply an extension of (2.2.7). In Section 2.2 we showed that $L^{\ddagger}_1(\theta) \in B^{\ddagger}(M; 1)$; here we do not hope for such a big dividend. It seems reasonable (in view of Theorem 4.1) to expect that $L^{\ddagger}_n(\theta) \in B^{\ddagger}(M; \delta)$ for some $n \geq 2$ depending on δ. Also note that since $F(x)$ has an infinite second moment, we can no longer write $[1-F^{\ddagger}_{(1)}(\theta)]/(-\mu_1 i\theta)$ as $\mu_2 F^{\ddagger}_{(2)}(\theta)/2\mu_1^2$.

Before proceeding to study the integrability properties of $L_n(x)$, we derive an interesting alternate expression for $L^{\ddagger}_n(\theta)$. Recall that part (a) of Theorem 2.1 deals with Taylor expansions of characteristic functions when moments of non-integral order are known to exist. Using this result (and setting $N = n+1$ for simplicity) we obtain

$$[1-F^{\ddagger}_{(1)}(\theta)]^N = \frac{|\theta|^{\delta N-1}[s^{\dagger}(\theta)]^N}{\mu_1^N} \exp\left\{i\pi N - (\operatorname{sgn}\theta)\left[\frac{(N-1)i\pi}{2} - Nic\right]\right\}$$

Choose n so that N is even and $\delta N = 1$. (This can be accomplished without loss of generality; if $\delta = \frac{1}{N} + \varepsilon$ for some small $\varepsilon > 0$, then replace the moment function $M(x)$ by $M'(x) = x^{\varepsilon}M(x)$.) Since c can be any real constant, take $c = (N-1)\pi/2N$. Then $e^{i\pi N} = 1$ and

$$\frac{[1-F^{\ddagger}_{(1)}(\theta)]^N}{-i\theta} = \frac{[s^{\dagger}(\theta)]^N}{\mu_1^N},$$

so that

$$L_n^{\ddagger}(\theta) = \frac{[s^{\dagger}(\theta)]^{n+1}}{\mu_1^{n+2}\ F_{(1)}^{\ddagger}(\theta)} \quad . \tag{4.2.3}$$

(4.2.3) provides a particularly transparent representation for $L_n^{\ddagger}(\theta)$, in addition to relating n and δ in a surprising simple manner. However since it is easier to deal with $F_{(1)}(x)$ than $s(x)$ we shall take (4.2.2) rather than (4.2.3) as our starting point.

Because we are now assuming that $F(x)$ has an infinite variance, the moment function $M(x)$ must necessarily grow slowly, for example, like

$$x^{\frac{1-\delta}{2}} \log x.$$

Generally speaking, $x^{1-\delta}/M(x)$ cannot be bounded; otherwise

$$\int_0^\infty x^2 dF(x) = \int_0^\infty x^{1+\delta} M(x) \left[\frac{x^{1-\delta}}{M(x)}\right] dF(x) < \infty.$$

For convenience we shall assume (throughout this section only) that $x^{1-\delta}/M(x)$ is, in addition, a nondecreasing function.

Previous studies of the infinite variance case have yielded "o" or "O" type estimates of the remainder function $L_n(x)$. Here our goal is somewhat more ambitious; we want to show that $L_n(x)$ belongs to some appropriate moment class, and for this purpose we shall utilize the approach presented in Chapter 2. In other words, we shall attempt to develop some additional "smoothing magic" in order to cope with transforms such as (4.2.2)

Consider once more the remainder $L_1(x)$, assuming now that $F(x) \in \mathcal{D}(M;\ 1+\delta)$. Since $M(x)$ is nondecreasing,

$$\int_0^x u^{\delta-1}\, M(u)du \le \frac{x^\delta M(x)}{\delta} ,$$

and therefore

$$\int_0^\infty \int_0^x u^{\delta-1}\, M(u)du dF_{(1)}(x) < \infty.$$

By Fubini's theorem it follows that

$$\text{(4.2.4)} \qquad \int_0^\infty x^{\delta-1}\, M(x)[1-F_{(1)}(x)]dx < \infty.$$

(4.2.4) is equivalent to writing

$$\sum_{n=1}^\infty \int_{2^{\delta(n-1)}}^{2^{\delta n}} x^{\delta-1}\, M(x)[1-F_{(1)}(x)]dx < \infty,$$

and using the monotonicity of $M(x)$ and $F_{(1)}(x)$ we can deduce the absolute convergence of the series

$$\sum_{n=1}^\infty 2^{n\delta^2}\, M(2^{\delta(n-1)})F^c_{(1)}(2^{\delta n}).$$

Consequently

$$\sum_{n=1}^\infty 2^{2n\delta^2} \left\{ M(2^{\delta(n-1)})\, F^c_{(1)}(2^{\delta n})\right\}^2 < \infty,$$

and this, in turn, implies that

$$\sum_{n=1}^\infty \int_{2^{\delta n}}^{2^{\delta(n+1)}} x^{2\delta-1} \left\{ M(x)\; F^c_{(1)}(x)\right\}^2 dx < \infty.$$

Therefore

$$\int_0^\infty x^{2\delta-1} \left\{ M(x)\; F^c_{(1)}(x)\right\}^2 dx < \infty,$$

and since $M(x) = 0(M(\frac{x}{2}))$, we can conclude that

$$(4.2.5) \qquad \left[F^c_{(1)}(\tfrac{x}{2}) \right]^2 \in L(M;\ 2\delta-1)$$

and

$$(4.2.6) \qquad F^c_{(1)}(\tfrac{x}{2})\ F^c_{(1)}(x) \in L(M;\ 2\delta-1).$$

Furthermore, since $x^{1-\delta}/M(x)$ is nondecreasing,

$$\int_0^\infty x^{2\delta-1}\ [M(x)]^2 \left\{ \int_0^{x/2} [F^c_{(1)}(x-z) - F^c_{(1)}(x)]\ f_{(1)}(z)dz \right\} dx$$

$$\leq \int_0^\infty x^{2\delta-1}\ [M(x)]^2 \int_0^{x/2} zf_{(1)}(x-z)f_{(1)}(z)dzdx$$

$$= \int_0^\infty x^{2\delta-1}\ [M(x)]^2 \int_0^{x/2} \frac{z^{1-\delta}}{M(z)}\ f_{(1)}(x-z)z^\delta\ M(z)f_{(1)}\ (z)dzdx$$

$$\leq \text{(constant)} \int_0^\infty (\tfrac{x}{2})^\delta\ M(\tfrac{x}{2})\ f_{(1)}(\tfrac{x}{2}) \int_0^{x/2} z^\delta\ M(z)f_{(1)}(z)dzdx < \infty .$$

Therefore

$$(4.2.7) \qquad \int_0^{x/2} \left\{ F^c_{(1)}(x-z) - F^c_{(1)}(x) \right\} f_{(1)}(z)dz \in L(M;\ 2\delta-1).$$

(4.2.5), (4.2.6), and (4.2.7) together imply that

$$2F^c_{(1)}(x) - F^c_{(1)2}(x) \in L(M;\ 2\delta-1);$$

(cf. the derivation of (2.2.10) in Section 2.) Using Theorem 2.2 we can

then conclude that $L_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2\delta-1)$. However this result is not as strong as might be expected; clearly $2\delta - 1 < \delta$ for $0 < \delta < 1$ and furthermore, $2\delta - 1$ is negative unless $0 < \delta < \frac{1}{2}$.

Theorem 4.1 seems to suggest that if more terms are added to the expansion for $1/[1-F^{\ddagger}(\theta)]$ the remainder $L_n^{\ddagger}(\theta)$ will be a member of $\mathcal{B}^{\ddagger}(M; \delta)$ provided n (depending on δ) is large enough. Unfortunately the methods we have used to analyze $L_1^{\ddagger}(\theta)$ do not appear to be adequate for dealing with transforms of the type

$$(4.2.8) \qquad \frac{[1 - F_{(1)}^{\ddagger}(\theta)]^{n+1}}{- i\theta}$$

when $n \geq 2$. One apparent drawback is that $F(x)$ must possess a derivative of order $(n-1)$. Although it may be possible to overcome such obstacles by more refined analysis, we shall follow a different approach in the next section.

4.3 Two Auxiliary Functions: $\ell_1(x)$ and $\ell_2(x)$

Let us return to (4.2.2) and rewrite $L_n^{\ddagger}(\theta)$ as

$$(4.3.1) \qquad L_n^{\ddagger}(\theta) = \frac{[1-F_{(1)}^{\ddagger}(\theta)]}{\mu_1^n \, F_{(1)}^{\ddagger}(\theta)} \left\{\frac{1 - f_{(1)}^{\dagger}(\theta)}{(-i\theta)^{1/n}}\right\}^n .$$

Clearly neither $(-i\theta)^{-1/n}$ nor $f_{(1)}^{\dagger}(\theta)/(-i\theta)^{1/n}$ is the Fourier transform of an integrable function. However we *can* express their difference as the transform of the difference of two appropriately chosen integrable functions. This device will, in fact, enable us to avoid the difficulties mentioned above and obtain the desired analogue of Theorem 4.1.

First we need to introduce two auxiliary functions. For $x > 0$ define

$$\ell_1(x) = \frac{1-F_{(1)}(x)}{x^{1-\gamma}}$$

and
$$\ell_2(x) = \int_0^x \left\{\frac{1}{(x-y)^{1-\gamma}} - \frac{1}{x^{1-\gamma}}\right\} f_{(1)}(y)\,dy,$$

where $\gamma = 1/k$ and $k \geq 2$ is the smallest integer such that

$$x^{1/k} = O(x^{\delta} M(x)) \quad \text{as} \quad x \to \infty.$$

(Since we are dealing with positive random variables, we may take $\ell_1(x) = \ell_2(x) = 0$ for $x \leq 0$; otherwise this definition must be extended.) These ℓ-functions faintly resemble the g-functions which appear in the proof of Lemma 2 of Smith (1967; page 275).

Notice that

$$\ell_1(x) - \ell_2(x) = \frac{1}{x^{1-\gamma}} - \int_0^x \frac{f_{(1)}(y)}{(x-y)^{1-\gamma}}\,dy, \tag{4.3.2}$$

and this difference is integrable, since both $\ell_1(x)$ and $\ell_2(x)$ belong to $L(I;0)$. (We shall not prove that $\ell_1(x)$ and $\ell_2(x)$ are integrable, since this is a consequence of Lemmas 4.2 and 4.4 given below.) Consequently the Fourier transform of (4.3.2) exists and is given by

$$\Gamma(\gamma)\left[\frac{1-f^{\dagger}_{(1)}(\theta)}{(-i\theta)^{\gamma}}\right],$$

so that the expression in braces on the right-hand side of (4.3.1) corresponds to the n-fold convolution

$$[\ell_1(x) - \ell_2(x)]^{*n}\ .$$

Therefore we may deal with (4.3.1) by performing a binomial expansion and studying the integrability properties of convolutions of the type

$$[\ell_1^{*j}(x)] * [\ell_2^{*(k-j)}(x)]$$

for $j = 0,1,\ldots,k$, where ℓ_i^{*m} denotes the mth iterated convolution of ℓ_i with itself.

Before proceeding to the main result of this chapter, we establish the following four lemmas:

<u>LEMMA 4.2</u> *If* $F(x) \in \mathcal{D}(M;\, 1+\delta)$ *for* $0 < \delta < 1$ *and* $M(x) \in M^*$, *then* $\ell_1(x) \in L(M;\, \delta-\gamma)$.

<u>PROOF</u>. The symbol K will be used (here and throughout this section) to denote a generic positive constant. Theorem 2.1 implies that $f_{(1)}(x) \in L(M;\, \delta)$; consequently by Fubini's theorem

$$\int_0^\infty \frac{F_{(1)}^c(x)\, x^{\delta-\gamma}}{x^{1-\gamma}}\, M(x)dx = \int_0^\infty x^{\delta-1} M(x) \int_x^\infty f_{(1)}(u)dudx$$

$$= \int_0^\infty \int_0^u x^{\delta-1}\, M(x) f_{(1)}(u)dxdu \le K \int_0^\infty u^\delta\, M(u) f_{(1)}(u)du < 0 .$$

<u>LEMMA 4.3</u> *If* $F(x) \in \mathcal{D}(M;\, 1+\delta)$ *for* $0 < \delta < 1$ *and* $M(x) \in M^*$, *then* for all $z \ge 0$,

$$\int_{2z}^\infty x^\delta\, M(x)\,|\ell_1(x-z)-\ell_1(x)|\,dx < K\, z^\gamma.$$

PROOF. Clearly for $0 \le z \le x/2$

$$|\ell_1(x-z) - \ell_1(x)| = \Psi_1(x,z) + \Psi_2(x,z),$$

say, where

$$\Psi_1(x,z) = F^c_{(1)}(x)\left\{\frac{1}{(x-z)^{1-\gamma}} - \frac{1}{x^{1-\gamma}}\right\}$$

and $\Psi_2(x,z) = \dfrac{1}{(x-z)^{1-\gamma}} \displaystyle\int_{x-z}^{x} f_{(1)}(y)dy.$

Since

$$\Psi_1(x,z) < \frac{KF^c_{(1)}(x)z}{x^{1-\gamma}} = \frac{K\ell_1(x)z}{x},$$

it follows by Lemma 4.2 that

$$\int_{2z}^{\infty} x^{\delta}M(x)\ \Psi_1(x,z)dx < K\int_{2z}^{\infty} zx^{\delta-1}\ M(x)\ell_1(x)dx$$

$$< Kz^{\gamma}\int_{2z}^{\infty} x^{\delta-\gamma}\ M(x)\ell_1(x)dx < Kz^{\gamma}.$$

Similarly since

$$\Psi_2(x,z) < \frac{K\ f_{(1)}(\frac{x}{2})z}{x^{1-\gamma}},$$

Lemma 4.2 and the fact that $M(2x) = O(M(x))$ imply that

$$\int_{2z}^{\infty} x^{\delta}M(x)\ \Psi_2(x,z)dx < K\int_{2z}^{\infty} zx^{\delta+\gamma-1}\ M(x)f_{(1)}(\tfrac{x}{2})dx$$

$$< Kz^{\gamma}\int_{2z}^{\infty} x^{\delta}M(x)f_{(1)}(\tfrac{x}{2})dx < Kz^{\gamma}.$$

The next result is included for the sake of completeness, since it parallels Lemma 4.2. Actually we shall only need to use the fact that $\ell_2(x) \in L(I;\ 0)$.

LEMMA 4.4 *Suppose* $F(x) \in \mathcal{D}(M;\ 1+\delta)$ for $0 < \delta < 1$ and $M(x) \in M^*$. *Furthermore assume that there exists some positive constant* $\varepsilon < 1 - \delta$ *such that* $y^{\varepsilon}/M(y)$ *is increasing. Then* $\ell_2(x) \in L(M;\ \delta-\gamma)$.

PROOF. $\ell_2(x) = \Lambda_1(x) + \Lambda_2(x)$, where

$$\Lambda_1(x) = \int_0^{x/2} \left\{ \frac{1}{(x-y)^{1-\gamma}} - \frac{1}{x^{1-\gamma}} \right\} f_{(1)}(y)\,dy$$

and
$$\Lambda_2(x) = \int_{x/2}^{x} \left\{ \frac{1}{(x-y)^{1-\gamma}} - \frac{1}{x^{1-\gamma}} \right\} f_{(1)}(y)\,dy .$$

Since $f_{(1)}(x)$ is nonincreasing, it is not difficult to see that

$$\Lambda_2(x) \le K\, f_{(1)}\left(\frac{x}{2}\right) x^{\gamma} .$$

Therefore (using the fact that $M(2x) = O(M(x))$,)

$$\int_0^{\infty} x^{\delta-\gamma} M(x)\Lambda_2(x)\,dx \le K \int_0^{\infty} x^{\delta} M(x) f_{(1)}\left(\frac{x}{2}\right) dx < \infty .$$

For $0 \le y \le x/2$,

$$\frac{1}{(x-y)^{1-\gamma}} - \frac{1}{x^{1-\gamma}} = \frac{1}{x^{1-\gamma}} \left[\frac{1}{(1-\frac{y}{x})^{1-\gamma}} - 1 \right]$$

$$= \frac{1}{x^{1-\gamma}} \sum_{k=1}^{\infty} \begin{bmatrix} k-\gamma \\ k \end{bmatrix} \left(\frac{y}{x}\right)^k \le K \left(\frac{y}{x^{2-\gamma}} \right).$$

Consequently by Fubini's theorem

$$\int_0^\infty x^{\delta-\gamma} M(x)\Lambda_1(x)\,dx \le K \int_0^\infty x^{\delta-\gamma} M(x) \int_0^{x/2} \frac{y}{x^{2-\gamma}} f_{(1)}(y)\,dy\,dx$$

$$= K \int_0^\infty x^{\delta-2} M(x) \int_0^{x/2} y\, f_{(1)}(y)\,dy\,dx$$

$$= K \int_0^\infty y\, f_{(1)}(y) \int_{2y}^\infty x^{\delta-2} M(x)\,dx\,dy$$

$$\le K \int_0^\infty y^{1-\varepsilon} f_{(1)}(y) M(2y) \int_{2y}^\infty x^{\delta-2+\varepsilon}\,dx\,dy$$

$$\le K \int_0^\infty y^{\delta} M(2y) f_{(1)}(y)\,dy < \infty$$

Therefore both $\Lambda_1(x)$ and $\Lambda_2(x)$ belong to $L(M;\ \delta-\gamma)$, proving the lemma.

LEMMA 4.5 *Suppose* $F(x) \in \mathcal{D}(M;\ 1+\delta)$ for $0 < \delta < 1$ and $M(x) \in M^*$. *Furthermore assume that there exists some positive constant* $\varepsilon < 1-\delta$ *such that* $y^{\varepsilon}/M(y)$ *is increasing. Then for all* $z \ge 0$,

$$\int_{2z}^\infty x^{\delta} M(x)\,|\ell_2(x-z) - \ell_2(x)|\,dx < K\, z^{\gamma}.$$

PROOF. It is easily verified that

$$|\ell_2(x-z) - \ell_2(z)| \le J_1(x) + J_2(x),$$

where

$$J_1(x) = \int_0^{x-z} \left\{ \frac{1}{(x-z-y)^{1-\gamma}} - \frac{1}{(x-z)^{1-\gamma}} - \frac{1}{(x-y)^{1-\gamma}} + \frac{1}{x^{1-\gamma}} \right\} f_{(1)}(y)dy$$

and $\quad J_2(x) = \int_{x-z}^{x} \left\{ \frac{1}{(x-y)^{1-\gamma}} - \frac{1}{x^{1-\gamma}} \right\} f_{(1)}(y)\,dy.$

Since $f_{(1)}(x)$ is decreasing and $M(2x) = O(M(x))$,

$$\int_0^\infty x^\delta M(x) J_2(x)\,dx \le \int_0^\infty x^\delta M(x) f_{(1)}(\tfrac{x}{2}) \left[\frac{z^\gamma}{\gamma} - \frac{z}{x^{1-\gamma}} \right] dx \le K z^\gamma.$$

It is more difficult to deal with $J_1(x)$: For $0 < \rho < 1/2$, $\lambda = 1 - \rho$, and $z \le x/2$,

$$J_{11}(x) = \int_{\rho x}^{x-z} \left\{ \frac{1}{(x-z-y)^{1-\gamma}} - \frac{1}{(x-z)^{1-\gamma}} - \frac{1}{(x-y)^{1-\gamma}} + \frac{1}{x^{1-\gamma}} \right\} f_{(1)}(y)dy$$

$$\le f_{(1)}(\rho x) \left[\frac{(\lambda x-z)^\gamma}{\gamma} - \frac{(\lambda x)^\gamma}{\gamma} + \frac{z^\gamma}{\gamma} + (\lambda x - z)x^{\gamma-1} - \frac{\lambda x - z}{(x-z)^{1-\gamma}} \right].$$

We can write

$$(\lambda x - z)^\gamma - (\lambda x)^\gamma = (\lambda x)^\gamma \sum_{j=1}^{\infty} \binom{\gamma}{j} (-1)^j \left(\frac{z}{\lambda x}\right)^j ,$$

where the series converges absolutely, since $|z/\lambda x| < 1$ for $z \le x/2$. Therefore

$$|(\lambda x - z)^\gamma - (\lambda x)^\gamma| \le K z x^{\gamma-1} .$$

Likewise

$$(\lambda x - z)x^{\gamma-1} - \frac{\lambda x - z}{(x-z)^{1-\gamma}} = (z - \lambda x)x^{\gamma-1} \sum_{j=1}^{\infty} \binom{\gamma+j-1}{j} \left(\frac{z}{x}\right)^j ,$$

so that

$$|(\lambda x-z)x^{\gamma-1} - \frac{\lambda x-z}{(x-z)^{1-\gamma}}| \leq K\ zx^{\gamma-1}.$$

It is not hard to show that $f_{(1)}(\varepsilon x) \in L(M;\ \delta)$ for $M(x) \in M^*$; consequently, it follows from the above that

$$\int_0^\infty x^\delta\ M(x)J_{11}(x)dx < K\ z^\gamma.$$

Now let $J_{12}(x) = J_1(x) - J_{11}(x)$. Over the range of integration $0 < y > \rho x$, we can write

$$\frac{1}{(x-z-y)^{1-\gamma}} - \frac{1}{(x-z)^{1-\gamma}} - \frac{1}{(x-y)^{1-\gamma}} + \frac{1}{x^{1-\gamma}} =$$

$$= x^{\gamma-1} \sum_{j=2}^{\infty} \binom{j-\gamma}{j} \left[\left(\frac{z+y}{x}\right)^j - \left(\frac{z}{x}\right)^j - \left(\frac{y}{x}\right)^j\right]$$

$$= x^{\gamma-1} \sum_{j=2}^{\infty} \binom{j-\gamma}{j} x^{-j} \sum_{i=1}^{j-1} \binom{j}{i} z^i\ y^{j-i},$$

where the series converges uniformly in y, since $0 < \rho < 1/2$, and $(z+y)/x < \rho + 1/2 < 1$. Therefore term-by-term integration of the series is permissible, and

$$J_{12}(x) = \sum_{j=2}^{\infty} \sum_{i=1}^{j-1} \binom{j-\gamma}{j} \binom{j}{i} x^{\gamma-1-j}\ z^i \int_0^{\rho x} y^{j-i}\ f_{(1)}(y)dy.$$

Let ν be a positive integer; by assumption

$$\int_0^\infty \int_0^{\rho x} y^\nu\ f_{(1)}(y)dy = \rho^{\delta-\nu} \int_0^\infty y^\nu\ f_{(1)}(y) \int_{y/\rho}^\infty x^{\delta-\nu-1}\ M(x)dx$$

$$= \rho^{\delta-\nu} \int_0^\infty y^\nu f_{(1)}(y) \int_{y/\rho}^\infty \left[\frac{M(x)}{x^\varepsilon}\right] x^{\delta-\nu-1+\varepsilon}\, dxdy$$

$$\le \rho^{\delta-\nu+\varepsilon} \int_0^\infty y^{\nu-\varepsilon} M(\tfrac{y}{\rho})\, f_{(1)}(y) \int_{y/\rho}^\infty x^{\delta-\nu-1+\varepsilon}\, dxdy$$

$$= \frac{1}{\nu-\varepsilon-\delta} \int_0^\infty y^\delta M(\tfrac{y}{\rho})\, f_{(1)}(y)dy < \infty.$$

In other words,

$$\rho^{\delta-\nu}(\nu-\varepsilon-\delta) \int_0^{\rho x} y^\nu f_{(1)}(y)dy \in L(M,\ \delta-\nu-1).$$

This implies that

$$\int_0^\infty x^\delta M(x) J_{12}(x)dx \le$$

$$\le z^\gamma \sum_{j=2}^\infty \sum_{i=1}^{j-1} \binom{j-\gamma}{j} \binom{j}{i} (\tfrac{1}{2})^{i-\gamma} \int_0^\infty x^{\delta+i-1-j} M(x) \int_0^{\rho x} y^{j-i} f_{(1)}(y)dy$$

$$\le K\, z^\gamma \sum_{j=2}^\infty \sum_{i=1}^{j-1} \binom{j-\gamma}{j} \binom{j}{i} (\tfrac{1}{2})^{i-\gamma} \frac{\rho^{j-i-\delta}}{j-i-\varepsilon-\delta}$$

$$\le \frac{K\, 2^\gamma \rho^{-\delta} z^\gamma}{1-\varepsilon-\gamma} \sum_{j=2}^\infty \binom{j-\gamma}{\gamma} \sum_{i=1}^{j-1} \binom{j}{i} (\tfrac{1}{2})^i \rho^{j-i}$$

$$= \frac{K\, 2^\gamma \rho^{-\delta} z^\gamma}{1-\varepsilon-\delta} \sum_{j=2}^\infty \binom{j-\gamma}{\gamma} \left[(\tfrac{1}{2}+\rho)^j - (\tfrac{1}{2})^j - \rho^j\right]$$

$$= \frac{K\, 2^\gamma \rho^{-\delta} z^\gamma}{1-\varepsilon-\delta} \left[\frac{1}{(1-\frac{1}{2}\rho)^{1-\gamma}} - \frac{1}{(1-\rho)^{1-\gamma}} - \frac{1}{(\frac{1}{2})^{1-\gamma}} - 1\right]$$

$$< K\, z^\gamma .$$

Therefore

$$\int_0^\infty x^\delta \, M(x) J_{12}(x) dx \; < \; K \, z^\gamma ,$$

completing the proof of the lemma.

For $x \geq 0$ define $R(x) = U(x) - F_{(1)}(x)$ and

$$S(x) = \int_0^x \frac{1 - F_{(1)}(x)}{\mu_1} \, dx.$$

(Take $R(x) = S(x) = 0$ for $x < 0$.) Clearly $R^{\ddagger}(\theta) = 1 - F^{\ddagger}_{(1)}(\theta)$. $S(x)$ however, is an increasing function of unbounded variation, (since the variance of $F(x)$ is infinite), and therefore does not possess a Fourier-Stieltjes transform. Even so we may formally identify $S(x)$ with the quotient $[1 - F^{\ddagger}_{(1)}(\theta)]/(-\mu_1 i\theta)$. $R(x)$ and $S(x)$ play a role in the following representation for the renewal function:

<u>THEOREM 4.6</u> *Suppose that* $\{X_n\}_{n=1}^{\infty}$ *is a sequence of iid positive random variables with distribution function* $F(x) \in \mathcal{D}(M; 1+\delta) \cap C$ *for* $0 < \delta < 1$ *and* $M(x) \in M^*$ *such that*

$$\int_0^\infty x^2 \, dF(x) \; = \; + \infty$$

Suppose, furthermore, that there exists some positive constant $\varepsilon < 1 - \delta$ *such that* $y^\varepsilon/M(y)$ *is increasing. Let* $k \geq 2$ *be the smallest integer such that* $x^{1/k} = O(x^\delta M(x))$ *as* $x \to \infty$. *Then*

$$H(x) = \sum_{n=1}^{\infty} P\{x_1 + \ldots + x_n \le x\} =$$

(4.3.3)

$$= \left(\frac{x}{\mu_1} - 1\right) U(x) + \sum_{j=1}^{k} S(x)*[R(x)]^{*(j-1)} + L_k(x),$$

where $L_k(x)$ *is a function of bounded variation in the class* $\mathcal{B}(M;\ \delta)$ *and, in particular,*

$$L_k(x) = o\left(\frac{1}{x^{\delta} M(x)}\right)$$

as $x \to \infty$.

N.B. Theorem 4.6 does not cover the subclass of M^* consisting of functions $M(x)$ which grow "almost" as fast as a fractional power of x. For example, choose $\delta = 1/2$ and let $M(x)$ grow asymptotically like $x^{1/2}/\log x$. Then the variance of $F(x)$ can be infinite, but there exists no constant $\varepsilon < 1/2$ such that $y^{\varepsilon - \frac{1}{2}}/\log y$ is increasing. This "borderline" situation can be dealt with, although the details would require extensive digression; we refer the reader to the discussion of the subclass $M_3(\rho)$ in the paper by Smith (1967).

Secondly we note that *both* δ and $M(x)$ have a bearing on the size of k. We implied in Section 4.2 that k should be chosen as the smallest positive integer such that $1/k \le \delta$, and this choice will, in fact yield a valid expansion. However $M(x)$ might conceivably by asymptotically equal to (say) $x^{1/4} \log x$, and in such cases it is possible to reduce the number of terms used in the expansion.

Finally, it should be pointed out that when $\delta = 1$ expression (4.3.3) reduces to the expansion given in Theorem 2.4.

PROOF OF THEOREM 4.6 As we have noted in Section 2.2, the renewal function $H(x)$ does not possess a Fourier-Stieltjes transform. Consequently we shall study the transform of the modified renewal function $H_\zeta(x)$ defined by (2.2.1) for $0 < \zeta < 1$. By Theorem 2.1

$$1 - \zeta F^{\ddagger}(\theta) = (1-\zeta) - \mu_1 \zeta i\theta - \zeta|\theta|^{1+\delta} s^{\dagger}(\theta),$$

where $s^{\dagger}(\theta) \in L^{\dagger}(I; 0)$ and $s^{\dagger}(\theta) = 0$. Writing (as before) $\beta = (1-\zeta) - \mu_1 i\theta$, it is not difficult to show that

$$\left|\frac{\theta}{\beta}\right|^2 = \frac{\theta^2}{(1-\zeta)^2 + \zeta^2\mu_1^2\theta^2},$$

and consequently $|\theta| = O(|\beta|)$ for all θ in a small open interval I centered about the origin, *uniformly with respect to* ζ. Together these facts imply that if $\theta \in I$, then

$$\left|H_\zeta^{\dagger}(\theta) - \frac{\Delta_a^{\dagger}(\theta)}{\beta}\right| = O(1)$$

uniformly for $0 < \zeta < 1$. (Recall that $\Delta_a^{\dagger}(\theta)$ is the characteristic function of the triangular density $\Delta_a(x)$.) Defining $H(x)$ and $\overline{H}(x)$ as in Section 2.2 no additional changes in the proof of Theorem 4.2 are needed to conclude that

$$(4.3.4) \qquad H(x) - \overline{H}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\, \Delta_a^{\dagger}(\theta)\left\{\frac{1}{1-F^{\ddagger}(\theta)} - \left[\frac{1}{-\mu_1 i\theta} + 1\right]\right\} d\theta .$$

However we now use the formal expansion (4.2.1) as opposed to (2.2.7), where the number of terms is chosen to be k as defined in the statement of the theorem. Our goal will be to demonstrate that the remainder term $L_k^{\ddagger}(\theta)$ given by (4.3.1) is a member of the class $B^{\ddagger}(M; \delta)$. (Since the many-valued function z^α for non-integer values of α occurs here, we

establish the following convention to avoid ambiguity: Slit the complex plane along the negative real axis from 0 to $-\infty$, and define z^{α} in the open slit plane when $|\arg z| < \pi$ by analytic continuation from the positive real axis.)

As before write $\gamma = 1/k$. Since $F_{(1)}(x)$ is absolutely continuous with density $f_{(1)}(x)$,

$$\text{(4.3.5)} \quad \Delta_a^{\dagger}(\theta)L_k^{\ddagger}(\theta) = \frac{\Delta_a^{\dagger}(\theta)[1-f_{(1)}^{\dagger}(\theta)]}{[\mu_1\Gamma(\gamma)]^k}\left\{\frac{\Gamma(\gamma)}{(-i\theta)^{\gamma}} - \frac{\Gamma(\gamma)f_{(1)}^{\dagger}(\theta)}{(-i\theta)^{\gamma}}\right\}^k .$$

We recognize the right-hand side of (4.3.5) as the Fourier transform of a constant times

$$\Delta_a(x)*[\ell_1(x)-\ell_2(x)]^{*k} - \Delta_a(x)*f_{(1)}(x)*[\ell_1(x)-\ell_2(x)]^{*k}$$

$$= \Delta_a(x)* \sum_{j=0}^{k} \binom{k}{j}(-1)^j\left[\ell_1^{*j}(x)*\ell_2^{*(k-j)}(x) - f_{(1)}(x)*\ell_1^{*j}(x)*\ell_2^{*(k-j)}(x)\right].$$

For convenience denote the k-fold convolution $\ell_1^{*j}(x)*\ell_2^{*(k-j)}(x)$ as $\ell^{*k}(x)$. Then it will suffice to prove

$$\ell^{*k}(x) - f_{(1)}(x)*\ell^{*k}(x) \in L(M;\delta).$$

The convolution $\ell^{*k}(x)$ can be rewritten as the multiple integral

$$\text{(4.3.6)} \quad \int\!\!\cdots\!\!\int_{u^* \le x} \ell_{i_1}(x-u^*)\ell_{i_2}(u_1)\ell_{i_3}(u_2)\cdots \ell_{i_k}(u_{k-1})du_1 \cdots du_{k-1}$$

where $u^* = u_1 + \ldots + u_{k-1}$ and i_j can be either 1 or 2 for $j = 1, \ldots, k$. Without affecting the value of the integral itself, the range of integration for (4.3.6) can be replaced by the union of k sets,

$$S_x \cup T_1 \cup T_2 \cup \ldots \cup T_{k-1},$$

where S_x is the set of $(k-1)$ - dimensional points $(u_1,\ldots,u_{k-1})$ such that

$$u^* + \max_{1\le j\le k-1} u_j \le x \quad \text{and} \quad u^* < x,$$

and for $i = 1,\ldots,k-1$, T_i is the set of $(k-1)$ - dimensional points $(u_1,\ldots,u_{k-1})$ such that

$$u^* + \max_{1\le j\le k-1} u_j > x, \ u^* \le x, \ \text{and} \ u_i = \max_{1\le j\le k-1} u_j .$$

(Notice that although the sets T_i are not disjoint, their intersections form sets of measure zero.)

Consider a typical set T_i, say T_1. If $(u_1,\ldots,u_{k-1}) \in T_1$, then

$$0 \le x - u^* < u_1. \tag{4.3.7}$$

Set $v_1 = x-u^*$ and $v_j = u_j$ for $j = 2,\ldots,k-1$. The inequality (4.3.7) is equivalent to

$$\max(x-u^*, u_2,\ldots,u_{k-1}) \le u_1,$$

or, in other words

$$\sum_{i=1}^{k-1} v_j + \max_{1\le j\le k-1} v_j \le x.$$

Therefore

$$\int_{T_1}\cdots\int \ell_{i_1}(x-u^*)\ell_{i_2}(u_1)\ldots \ell_{i_k}(u_{k-1})du_1 \ldots du_{k-1}$$

$$= \int_{S_x}\cdots\int \ell_{i_1}(v_1)\ell_{i_2}\left(x - \sum_{j=1}^{k-1} v_j\right)\ell_{i_3}(v_2) \ldots \ell_{i_k}(v_{k-1})dv_1 \ldots dv_1. \tag{4.3.8}$$

Consequently we may write

$$(4.3.9) \qquad \ell^{*k}(x) = \sum_{j=1}^{k} W_j(x),$$

where

$$W_1(x) = \int \cdots \int_{S_x} \ell_{i_1}(x-u^*)\ell_{i_2}(u_1) \cdots \ell_{i_k}(u_{k-1})du_1 \cdots du_{k-1},$$

$$W_2(x) = \int \cdots \int_{S_x} \ell_{i_1}(u_1)\ell_{i_2}(x-u^*)\ell_{i_3}(u_2) \cdots \ell_{i_k}(u_{k-1})du_1 \cdots du_k$$

and so forth.

If $(u_1,\ldots,u_{k-1}) \in S_x$, then

$$x - u^* \geq u_i$$

for $i = 1,2,\ldots,k-1$. Therefore (by adding all $(k-1)$ inequalities)

$$(k-1)x - (k-1)u^* \geq u_i$$

and this implies

$$x - u^* \geq \frac{x}{k} .$$

Consider now a typical integral $W_j(x)$, say $W_1(x)$, and the corresponding integral

$$(4.3.10) \qquad \int_o^{x/2k} \left\{W_1(x-z)-W_1(x)\right\} f_{(1)}(z)dz .$$

Clearly we can rewrite

$$W_1(x-z) - W_1(x) = \Gamma_1(x,z) - \Gamma_2(x,z),$$

where

$$\Gamma_1(x,z)=\int_{u^*+\max\limits_{1\le j\le k-1} u_j \le x-j}\cdots\int \left[\ell_{i_1}((x-z)-u^*)-\ell_{i_1}(x-u^*)\right]\ell_{i_2}(u_1)\ldots\ell_{i_k}(u_{k-1})\,du_1\ldots du_{k-1}$$

and $$\Gamma_2(x,z) = \int_{x-z\le u^*+\max\limits_{1\le j\le k-1} u_j \le x}\cdots\int \ell_{i_1}(x-u^*)\ell_{i_2}(u_1)\ldots\ell_{i_k}(u_{k-1})\,du_1\ldots du_{k-1}.$$

By Fubini's theorem (using χ to denote the indicator function)

$$\int_0^\infty x^\delta\, M(x)\Gamma_1(x,z)\,dx =$$

$$= \int_{u_j\ge 0}\cdots\int \left\{\int_0^\infty x^\delta M(x)\left[\ell_{i_1}\left((x-z)-u^*\right)-\ell_{i_1}(x-u^*)\right]\cdot\right.$$

$$\left.\cdot\chi\left\{u^*+\max_{1\le j\le k-1} u_j \le x-z\right\}dx\right\}\ell_{i_2}(u_1)\ \ldots\ \ell_{i_k}(u_{k-1})\,du_1\ \ldots\ du_k$$

$$(4.3.11)\qquad = \int_{u_j\ge 0}\cdots\int \left\{\int_0^\infty (y+u^*)^\delta\, M(y+u^*)\left[\ell_{i_1}(y-z) - \ell_{i_1}(y)\right]\cdot\right.$$

$$\left.\chi\left\{\max_{1\le j\le k-1} u_j \le y-z\right\}dy\right\}\ell_{i_2}(u_1)\ \ldots\ \ell_{i_k}(u_{k-1})\,du_1\ \ldots\ du_{k-1}.$$

If $\max\limits_{1\le j\le k-1} u_j \le y-z$, then $\max\limits_{1\le j\le k-1} u_j \le y$, so that $u^* \le (k-1)y$.

Consequently on the range of integration for the integral in (4.3.11),

$$(y+u^*)^\delta\, M(y+u^*) \le K\, y^\delta\, M(y).$$

Furthermore, since we are working with $0 \le z \le x/2k$ (see (4.3.10),) we have

$$2kz \le y + u^* \le y + (k-1)y = ky.$$

Therefore we may apply Lemma 4.3 or 4.5 (depending on whether $i_1 = 1$ or $i_1 = 2$) and conclude that the inner integral in (4.3.11) is dominated by Kz^γ. On the other hand,

$$\int_{u_j \ge 0} \cdots \int \ell_{i_2}(u_1) \ \cdots \ \ell_{i_k}(u_{k-1})du_1 \ \cdots \ du_{k-1} < \infty,$$

since both $\ell_1(x)$ and $\ell_2(x)$ are integrable. It follows that (for $0 \le z \le x/2k$,)

$$\int_0^\infty x^\delta M(x)\Gamma_1(x,z)dx < Kz^\gamma \le K(1 + z^\delta M(z)) \ .$$

Next we show that $\Gamma_2(x,z)$ satisfies a corresponding inequality: By Fubini's theorem (recalling that $z \le x/2k$)

$$\int_0^\infty x^\delta M(x)\Gamma_2(x,z)dx =$$

$$(4.3.12) \qquad = \int_{u_j \ge 0} \cdots \int \left\{ \int_{2kz}^\infty x^\delta M(x)\ell_{i_1}(x-u^*)\chi\{x-z<u^*+ \max_{1 \le j \le k-1} u_j < x\}dx \right\}$$

$$\cdot \ \ell_{i_2}(u_1) \ \cdots \ \ell_{i_k}(u_{k-1})du_1 \ \cdots \ du_{k-1} \ .$$

Let $y = x - u^*$. Then on the range of integration for the inner integral, $u^* \le (k-1)y$, so that

$$(u^*+y)^\delta M(u^*+y) \quad \le \quad Ky^\delta M(y).$$

Consequently the inner integral in (4.3.12) is dominated by

$$(4.3.13) \qquad K \int_{2kz-u^*}^{\infty} y^{\delta} M(y) \ell_{i_1}(y) \chi\{y-z< \max_{1\le j\le k-1} u_j < y\} dy \ .$$

Since $u^* < (k-1) \max_{1\le j\le k-1} u_j$, it follows that

$$y > 2kz - (k-1) \max_{1\le j\le k-1} u_j$$

on the range of integration for (4.3.12). But $y < z + \max_{1\le j\le k-1} u_j$, so that

$$\max_{1\le j\le k-1} u_j > \frac{(2k-1)z}{k} \ ,$$

and for $k = 2,3,\ldots,$ we have

$$\max_{1\le j\le k-1} u_j > 3z/2 \ .$$

For $j = 2,\ldots,k-1$ set

$$g_{i_j}(x) = \ell_{i_j}(x) / \int_o^{\infty} \ell_{i_j}(u) du.$$

Let $U_{i_2}, U_{i_3}, \ldots, U_{i_{k-1}}$ be independent random variables with density functions $g_{i_2}(x), g_{i_3}(x), \ldots, g_{i_{k-1}}(x)$, respectively, and let $q(x)$ denote the density function of $\max_{2\le j\le k-1} U_{i_j}$. Thus (using (4.3.13)) it is easy to see that the multiple integral (4.3.12) is dominated by

$$(4.3.14) \qquad K \int_{3z/2}^{\infty} \left\{ \int_{w}^{w+z} y^{\delta} M(y) \ell_{i_1}(y) dy \right\} q(w) dw.$$

Now if $i_1 = 1$, then it is not difficult to verify that

$$(4.3.15) \qquad \ell_{i_1}(x) = 0\left(\frac{1}{x^{1+\delta-\gamma}\, M(x)}\right) \quad \text{as} \quad x \to \infty.$$

To show the same when $i_1 = 2$ write

$$\ell_2(x) = \Lambda_1(x) + \Lambda_2(x),$$

as in the proof of Lemma 4.2. Since $f_{(1)}(x) \in L(M, \delta)$ is nonincreasing and $M(x) \in M^*$,

$$(4.3.16) \qquad \begin{aligned} x^{1+\delta} M(x) f_{(1)}(x) &\le K \int_{x/2}^{x} u^{\delta} M(u) f_{(1)}(u)\,du \\ &\le K \int_{x/2}^{\infty} u^{\delta} M(u) f_{(1)}(u)\,du \to 0 \quad \text{as} \quad x \to \infty. \end{aligned}$$

Furthermore

$$(4.3.17) \qquad \Lambda_2(x) \le K\, x^{\delta} f_{(1)}\left(\tfrac{x}{2}\right),$$

and both (4.3.16) and (4.3.17) imply that

$$x^{1+\delta-\gamma}\, M(x)\Lambda_2(x) \to 0 \quad \text{as} \quad x \to \infty.$$

For $\Lambda_1(x)$ we can write

$$\Lambda_1(x) \le K\left\{\Lambda_{11}(x) + \Lambda_{12}(x)\right\},$$

say, where for fixed $\Delta > 0$,

$$\Lambda_{11}(x) = \frac{1}{x^{2-\gamma}} \int_{o}^{\Delta} y f_{(1)}(y)\,dy$$

and

$$\Lambda_{12}(x) = \frac{1}{x^{2-\gamma}} \int_\Delta^x y\ f_{(1)}(y)dy.$$

Clearly

$$x^{1+\delta-\gamma} M(x)\Lambda_{11}(x) \le \Delta x^{\delta-1} M(x) \to 0 \quad \text{as} \quad x \to \infty.$$

For $x > \Delta$,

$$x^{1+\delta-\gamma} M(x)\Lambda_{12}(x) = x^{\delta-1} M(x) \int_\Delta^x y^\delta M(y)\left\{\frac{y}{y^\delta M(y)}\right\} f_{(1)}(y)dy$$

$$\le \int_\Delta^\infty y^\delta M(y) f_{(1)}(y)dy,$$

which is bounded (and, in fact, can be made arbitrarily small by initially choosing Δ sufficiently large.) Thus (4.3.15) is valid for either $i_1 = 1$ or $i_1 = 2$.

Returning to the main argument, (4.3.15) can be used to show that (4.3.14) is dominated by

$$\int_{3z/2}^\infty \left\{ \int_w^{w+z} y^{\gamma-1}dy\right\} q(w)dw$$

$$\le K \int_{3z/2}^\infty \left\{(w+z)^\gamma - w^\gamma\right\} q(w)dw$$

$$\le K \int_{3z/2}^\infty w^\gamma\left(\frac{z}{w}\right) q(w)dw \le K\ z^\gamma \int_{3z/2}^\infty q(w)dw$$

$$\le K\ z^\gamma \le K(1 + z^\delta M(z)) \ .$$

Therefore we have shown that (for $0 \le z \le x/2k$),

$$\int_0^\infty x^\delta M(x)\Gamma_2(x,z)dx \le K(1 + z^\delta M(z)).$$

Now by Fubini's theorem,

$$\int_0^\infty x^\delta M(x) \int_0^{x/2k} |W_1(x-z)-W_1(x)| f_{(1)}(z)dzdx$$

$$= \int_0^\infty \left\{ \int_{2kz}^\infty x^\delta M(x)|W_1(x-z) - W_1(x)|dx\right\} f_{(1)}(z)dz$$

$$\le K \int_0^\infty \left\{1+z^\delta M(z)\right\} f_{(1)}(z)dz < \infty.$$

In other words, for $z \le x/2k$,

(4.3.18) $$\int_0^{x/2} [W_1(x-z) - W_1(x)]f_{(1)}(z)dz \in L(M; \delta).$$

(4.3.18) can be shown for $W_2(x)$, $W_3(x), \ldots,$ and $W_k(x)$. It follows from the representation (4.3.9) that

(4.3.19) $$\int_0^{x/2k} [\ell^{*k}(x-z) - \ell^{*k}(x)]f_{(1)}(z)dz \in L(M, \delta).$$

Recall that our goal is to demonstrate that

(4.3.20) $$\ell^{*k}(x) - \int_0^x \ell^{*k}(x-z)f_{(1)}(z)dz$$

$$= \ell^{*k}(x) \int_{x/2k}^\infty f_{(1)}(z)dz - \int_0^{x/2k}\left\{\ell^{*k}(x-z)-\ell^{*k}(x)\right\}f_{(1)}(z)dz$$

$$- \int_{x/2k}^x \ell^{*k}(x-z)f_{(1)}(z)dz$$

belongs to the class $L(M; \delta)$. Since

$$x^{\delta} M(x)[1 - F_{(1)}(x)] \to 0 \quad \text{as} \quad x \to \infty,$$

and $\ell^{*k}(x)$ is integrable,

$$\int_0^{\infty} x^{\delta} M(x) \ell^{*k}(x) \int_{/2k}^{\infty} f_{(1)}(z)\,dz\,dx < \infty. \tag{4.3.21}$$

Also,

$$\int_0^{\infty} x^{\delta} M(x) \int_{x/2k}^{x} \ell^{*}(x-z) f_{(1)}(z)\,dz\,dx$$

$$\leq K \int_0^{\infty} x^{\delta} M(x) f_{(1)}(x/2k) \int_0^{x} \ell^{*k}(u)\,du\,dx < \infty. \tag{4.3.22}$$

Together (4.3.19), (4.3.21), and (4.3.22) imply (4.3.20), and consequently

$$\frac{[1 - F^{\ddagger}_{(1)}(\theta)]^{k+1}}{-\mu_1 i\theta} \in \mathcal{B}^{\ddagger}(M; \delta). \tag{4.3.23}$$

This "smoothing magic" is in itself quite remarkable, since $[1 - F^{\ddagger}_{(1)}(\theta)] \in \mathcal{B}^{\ddagger}(M; \delta)$ and division by $-i\theta$ ordinarily entails the "loss" of one whole moment. Apparently the additional k-fold convolution in the numerator of (4.3.23) has the effect of making the "lost" moment "reappear"!

By introducing the special SMF $q^{\dagger}(\theta)$ and applying Theorem 2.2, we can use (4.3.23), (together with the fact that $F^{\ddagger}_{(1)}(\theta) \in \mathcal{B}^{\ddagger}(M; \delta)$) to conclude that

$$\frac{[1 - F^{\ddagger}_{(1)}(\theta)]^{k+1}}{-\mu_1 i\theta\, F^{\ddagger}_{(1)}(\theta)} \in \mathcal{B}^{\ddagger}(M;\, \delta),$$

and hence that $L_k^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M, \delta)$. We omit the details here, since the argument is precisely the same as given in Section 2.2 for $L^{\ddagger}(\theta)$. Note, however, that division by a function in the class $\mathcal{B}^{\ddagger}(M;\, \delta)$ precludes the possibility of showing that $L_k^{\ddagger}(\theta)$ is in a higher moment class, even if this could be demonstrated for the transform (4.3.23); in other words, it appears that we cannot improve on (4.3.24), at least via the Wiener-Pitt-Lévy-Smith approach.

We have shown that

$$\Delta_a^{\dagger}(\theta) \left\{\frac{1}{1-F^{\ddagger}(\theta)} - \left[\frac{1}{-\mu_1 i\theta} - 1\right]\right\}$$

appearing on the right-hand side of (4.3.4) is the Fourier transform of

$$\int_{-\infty}^{\infty} \Delta_a(x-z)\,d\left\{\sum_{j=1}^{k} R(z)* \,[U(z) - F_{(1)}(z)]^{*(j-1)} + L_k(z)\right\}$$

$$- \int_{-\infty}^{\infty} \Delta_a(x-z)\,dU(z) \;.$$

Proceeding as in Section 2.2 we then obtain the relationship

$$\int_{-\infty}^{\infty} \Delta_a(x-z)\,d\left\{\sum_{n=1}^{\infty} F_n(z)\right\} =$$

$$= \int_{-\infty}^{\infty} \Delta_a(x-z)\left\{U(z)\,\frac{dz}{\mu_1} + d\left[\sum_{j=1}^{k} R(z)*[U(z)-F_{(1)}(z)]^{*j-1}\right.\right.$$

$$\left.\left. + L_k(z) - U(z)\right]\right\} \;.$$

The result (4.3.3) follows by the "sandwiching" process used previously, together with a standard extension argument. As a by-product of our discussion it is evident that the series

$$\sum_{n=i}^{\infty} P\{X_1 + \ldots + X_n \leq x\}$$

is finite for every x.

CHAPTER V: HIGHER MOMENTS OF THE NUMBER OF RENEWALS

Let $\{X_n\}_{n=1}^{\infty}$ be the renewal process introduced in Chapter 2, and define N_t, the *number of renewals by time* t, as the largest integer k such that

$$X_1 + X_2 + \ldots + X_k \le t.$$

It is well known that all the moments of N_t are finite. The familiar renewal equation asserts that $H(t)$, the renewal function, is, in fact, the expectation of N_t. To show this let

$$Z_r = \begin{cases} 0 & \text{if } X_1 + \ldots + X_r \le t. \\ 1 & \text{if } X_1 + \ldots + X_r > t. \end{cases}$$

Then $N_t = \sum_{r=1}^{\infty} Z_r$, and

$$EN_t = \sum_{r=1}^{\infty} EZ_r = \sum_{r=1}^{\infty} P\{X_1 + \ldots + X_r \le t\} = H(t).$$

In Chapters 2 and 4 we dealt with the behavior of EN_t, the *first* moment of the number of renewals. We now extend our study to higher moments and, in particular, to the second and third cumulants of N_t.

5.1 Preliminary Background

W.L. Smith (1954) proved the following result for the variance of the number of renewals:

THEOREM 5.1 (Smith, 1954) *If* $F(x) \in \mathcal{D}(I; 2)$, *then as* $t \to \infty$,

$$\operatorname{Var} N_t = \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + o(t).$$

Theorem 5.1 is a consequence of the Key Renewal Theorem and the identity

$$EN_t^2 = H(t) + 2H(t)*H(t).$$

Apparently this type of argument does not generalize to higher moments. Using a different approach based on factorial moments and expansions for the Laplace-Stieltjes transform of $F(x)$, Smith (1959) showed that the nth cumulant of N_t has a *linear* asymptotic form:

THEOREM 5.2 (Smith, 1959) *If* $F(x) \in \mathcal{D}(I; n+p+1) \cap C$, $p \geq 0$, *then there exist constants* a_n *and* b_n *such that the* nth *cumulant of* N_t *is given by*

$$a_n t + b_n + \frac{\lambda(t)}{(1+t)^p}, \tag{5.1.1}$$

where $\lambda(t)$ *is a function of bounded variation, is* $o(1)$ *as* $t \to \infty$, *satisfies the condition*

$$\lambda(t) - \lambda(t-\alpha) = o(t^{-1})$$

as $t \to \infty$ *for every fixed* $\alpha > 0$, *and when* $p \geq 1$ *has the additional property that* $\lambda(t)/(1+t) \in L(I; 0)$.

The constants a_n and b_n are difficult to evaluate; Smith (1959) listed their values for $n = 1,2,\ldots,8$. a_n is a function of the first n moments of $F(x)$, whereas b_n is determined by the first $n+1$ moment. Since the numbers of renewals in adjacent intervals viewed over a long period of time are approximately independent, a cumulant of the

number of renewals in the entire time span is, roughly speaking, the sum of the cumulants of the numbers of renewals in the individual intervals. Consequently the asymptotic linearity of (5.1.1) is not surprising.

A familiar special case of this result is the following:

<u>COROLLARY 5.3</u> (Smith, 1959) *If* $F(x) \in \mathcal{D}(I; 3) \cap C$, *then as* $t \to \infty$,

$$\operatorname{Var} N_t = \frac{\mu_2 - \mu_1^2}{\mu_1^3}\, t + \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2} \right) + o(1) \quad .$$

An extension of Theorem 5.2 allowing for the existence of fairly general moments was obtained by Smith (1967):

<u>THEOREM 5.4</u> (Smith, 1967; page 271) *Suppose* $F(x) \in \mathcal{D}(M; \ell) \cap C$ *for some* $M(x) \in M^*$ *and some* $\ell > 1$. *Assume further that* $\mu_1 > 0$. *Then if* k *is the integer part of* ℓ, *there exist constants* $A_1(\ell), A_2(\ell), \ldots, A_k(\ell)$ *such that*

$$(5.1.2) \qquad \sum_{n=0}^{\infty} \binom{n+\ell-2}{n} P\{X_1 + \ldots + X_n \le x\} =$$

$$= \sum_{j=1}^{k} \frac{A_j(\ell)}{\Gamma(\ell-j+1)} \left(\frac{x}{\mu_1}\right)^{\ell-j} U(x) + \Lambda(x),$$

where $\Lambda(x) \in \mathcal{B}(M; 0)$.

Theorem 5.2 follows from Theorem 5.4 by using the fact that every cumulant of N_t can be written as a linear combination of sums like (5.1.2) .

Expansions for cumulants of N_t have been used for statistical analysis of the superposition of a relatively small number of renewal processes. As in Section 3.1 write $V_N(t)$ for the variance of the number of events occurring by time t in a superposition of N renewal processes. Cox and Smith (1954) originally proposed a variance-time curve analysis based on the expansion.

$$V_N(t) - \frac{N\sigma^2 t}{\mu_1^3} \sim N\left(\frac{1}{6} + \frac{\sigma^4}{2\mu_1^4} - \frac{\mu_3}{3\mu_1^3}\right),$$

as $t \to \infty$, where $\sigma^2 = \mu_2 - \mu_1^2$. $V_N(t)$ can be estimated from experimental observations of the superposition; see Cox and Smith (1953). By equating observed and theoretical values for the mean rate of occurrence and for the asumptotic slope and intercept, one can obtain three equations in four unknowns: N, μ_1, σ^2, and μ_3. Cox and Lewis (1966; page 215) suggest using the asymptotic slope of the *third* cumulant-time curve,

$$N\left(\frac{3\sigma^4}{\mu_1^5} - \frac{\mu_3}{\mu_1^4}\right),$$

to obtain a fourth equation.

We shall investigate the time-dependent behavior of the second and third cumulants of N_t in Sections 5.3 and 5.4. In Section 5.3 we shall prove a result (Theorem 5.8) which extends Corollary 5.3 in two directions, allowing for $F(x) \in \mathcal{D}(M; 3)_\cap C$ and replacing the remainder term "o(1)" by a known function and a much sharper remainder. A similar treatment for the third cumulant will be given in Section 5.4 However before pursuing these objectives we find it expedient to resume our discussion of smoothing magic.

5.2 More Smoothing Magic

In Sections 5.3 and 5.4 we shall encounter Fourier-Stieltjes transforms of the form

$$(5.2.1) \qquad \frac{A^{\dagger}(\theta)S^{\dagger}(\theta)}{-i\theta} ,$$

where $S(x)$ is absolutely continuous for $x > 0$, and both $A(x)$ and $S(x)$ vanish for $x < 0$ and belong to the class $\mathcal{B}(M; 1)$ for some $M(x) \in \mathcal{M}$. We have already dealt with a version of (5.2.1) in Section 2.2, where

$$A^{\dagger}(\theta) = S^{\dagger}(\theta) = 1 - F^{\dagger}_{(1)}(\theta).$$

In that situation we discovered via the smoothing magic of Lemma 2.3 that the convolution (5.2.1) is in the (unexpected) class $\mathcal{B}^{\dagger}(M; 1)$. We note that $\lim_{\theta\to 0} [1 - F^{\dagger}_{(1)}(\theta)] = 0$, or equivalently,

$$\int_{0-}^{\infty} d[U(x) - F_{(1)}(x)] = 0 .$$

If, more generally,

$$(5.2.2) \qquad \int_{0-}^{\infty} dA(x) = 0,$$

can we conclude that $A^{\dagger}(\theta)\; S^{\dagger}(\theta)/(-i\theta) \in \mathcal{B}^{\dagger}(M; 1)$? Theorem 5.5 provides an affirmative answer subject to a certain growth restriction on the right moment function $M(x)$.

THEOREM 5.5 *Let* $M(x) \in M$, *and suppose* $A(x)$ *and* $S(x)$ *are functions of bounded variation in the class* $B(\int_0^x M(u)du;\ 0)$, *such that both* $A(x)$ *and* $S(x)$ *vanish for* $x < 0$,

$$S(x) = \int_x^\infty s(u)du, \quad x \geq 0,$$

for some function $s(x) \in L(\int_0^x M(u)du;\ 0)$, *and*

$$\int_{0-}^\infty dA(x) = 0.$$

Then

$$\frac{A^\ddagger(\theta)\ S^\ddagger(\theta)}{-i\theta} \in B^\ddagger(\int_0^x M(u)du;\ 0)\ .$$

PROOF. For convenience we shall write

$$M_I(x) = \int_0^x M(u)du, \quad x \geq 0.$$

It is not hard to show that if $M(x) \in M$, then $M_I(x)$ is equivalent to a right moment function in the asymptotic sense mentioned in Section 2.1. Consequently $M_I(x) \in M$, and the moment classes in the statement of Theorem 5.5 are well-defined. We want to prove that

$$S(x)*A(x) = \int_0^x S(x-z)dA(z) \in L(M_I;\ 0).$$

We can write this convolution as the sum of three integrals,

$$I_1(x) + I_2(x) + I_3(x),$$

where

$$I_1(x) = \int_0^{x/2} \{S(x-z) - S(x)\}\, dA(z),$$

$$I_2(x) = \int_{x/2}^{x} S(x-z)\, dA(z),$$

and $$I_3(x) = -S(x) \int_{x/2}^{x} dA(z).$$

We shall deal with each integral separately.

Clearly

$$|I_1(x)| \le \int_0^{x/2} \int_{x-z}^{x} |s(y)|dy\ |dA(z)|.$$

By a change of variables followed by an interchange of order of integration,

$$\int_0^\infty \int_0^{x/2} \int_{x-z}^{x} M_I(x)|s(y)|dy|dA(z)|dx$$

$$= \int_0^\infty \int_0^{x/2} \int_0^{z} M_I(x)|s(y+x-z)|dy|dA(z)|dx$$

$$= \int_0^\infty \int_0^{v} \int_v^\infty M_I(u+v)|s(u+w)|dudw|dA(v)|.$$

Note that for $x_1 \ge 0$ and $x_2 \ge 0$,

$$M_I(x_1+x_2) = M_I(x_1) + \int_{x_1}^{x_1+x_2} M(u)du$$

$$\le M_I(x_1) + M(x_1)M_I(x_2)\ .$$

Consequently for $v \geq 0$,

$$\int_0^v \int_v^\infty M_I(u+v)\,|s(v+w)|\,dudw \leq$$

$$\int_0^v \int_v^\infty \{M_I(v-w) + M(v-w)M_I(u+w)\}\,|s(u+w)|\,dudw \;.$$

However

$$\int_0^v \int_v^\infty M_I(v-w)\,|s(u+w)|\,dudw \leq M_I(v) \int_0^v \int_v^\infty |s(u+w)|\,dudw$$

$$\leq M_I(v) \int_v^\infty u|s(u)|\,du = 0(M_I(v)) \quad \text{as} \quad v \to \infty.$$

Similarly

$$\int_0^v \int_v^\infty M(v-w)M_I(u+w)\,|s(u+w)|\,dudw$$

$$= \int_0^v \int_{v+w}^\infty M(v-w)M_I(z)\,|s(z)|\,dzdw$$

$$\leq \int_0^v \int_v^\infty M(v-w)M_I(z)\,|s(z)|\,dzdw$$

$$= \int_v^\infty \int_0^v M(v-w)M_I(z)\,|s(z)|\,dwdz$$

$$= M_I(v) \int_v^\infty M_I(z)\,|s(z)|\,dz = 0(M_I(v)) \quad \text{as} \quad v \to \infty.$$

Consequently $I_1(x) \in L(M_I;\, 0)$.

For the second integral we have

$$\int_0^\infty M_I(x) \int_{x/2}^{x} |S(x-z)| \; |dA(z)|dx$$

$$= \int_0^\infty \int_z^{2z} M_I(x)|S(x-z)|dx|dA(z)|$$

$$\leq \int_0^\infty \int_z^{2z} \{M_I(x-z) + M(x-z)M_I(z)\} \; |S(x-z)|dx|dA(z)|$$

$$= \int_0^\infty \int_z^{2z} M_I(x-z)|S(x-z)|dx|dA(z)|$$

$$+ \int_0^\infty \int_z^{2z} M(x-z)M_I(z)|S(x-z)|dx|dA(z)|$$

$$= \int_0^\infty \int_0^{z} M_I(x)|S(x)|dx|dA(z)|$$

$$+ \int_0^\infty M_I(z) \int_0^{z} M(x)|S(x)|dx|dA(z)|$$

$$\leq \int_0^\infty M_I(z)\{ \int_0^\infty |S(x)|dx\} \; |dA(z)|$$

$$+ \int_0^\infty M_I(z)\{ \int_0^\infty M(x)|S(x)|dx\} \; |dA(z)| \quad .$$

Since $\int_0^\infty \int_0^x M(u)du|s(x)|dx < \infty$, it follows by another application of Fubini's theorem for nonnegative functions that

$$\int_0^\infty M(u)|S(u)|du < \infty .$$

Consequently $I_2(x) \in L(M_I; 0)$.

Note that $A(x) \in B(I; 1)$, since $x \leq M_I(x)$ and $A(x) \in B(M_I(x); 0)$; therefore

$$(5.2.3) \qquad \int_x^\infty |dA(u)| = o(\tfrac{1}{x}) \quad \text{as} \quad x \to \infty.$$

Also, since $\dfrac{M_I(x)}{x} \leq M(x)$ and

$$\int_o^\infty M(u)|S(u)|du < \infty,$$

it follows that

$$(5.2.4) \qquad \int_o^\infty \frac{M_I(x)}{x} \; |S(x)|dx < \infty .$$

Combining (5.2.3) and (5.2.4) we get

$$\int_o^\infty \int_x^\infty |dA(u)|M_I(x)|S(x)|dx < \infty ,$$

which implies that $I_3(x) \in L(M_I; 0)$. This completes the proof of Theorem 5.5.

Suppose that the right moment function $M(x) \in M$ of Theorem 5.5 satisfies, in addition, the conditon

$$(5.2.5) \qquad M(2x) = 0(M(x)) \quad \text{as} \quad x \to \infty.$$

(Recall that this growth restriction was used in the definition of the class M^*.) Then for some constant $A > 0$,

$$\int_0^x M(u)\,du \geq \int_{x/2}^x M(u)\,du \geq \frac{x}{2} M(\tfrac{x}{2}) \geq \frac{x}{2}\frac{M(x)}{A} \quad .$$

Thus in the proof of Theorem 5.5 we can conclude that the integrals $I_1(x)$, $I_2(x)$, and $I_3(x)$ belong to the class $L(M;\ 1)$, yielding the following result:

<u>COROLLARY 5.6</u> *Let* $M(x) \in M^*$, *and suppose* $A(x)$ *and* $S(x)$ *are functions of bounded variation satisfying the following conditions:*

(A) $A(x)$ *and* $S(x)$ *vanish for* $x < 0$,

(B) For some function $s(x) \in L(M;\ 1)$,

$$S(x) = \int_x^\infty s(u)\,du, \quad x \geq 0,$$

(C) $A(x) \in B(M;\ 1)$,

and (D) $\displaystyle\int_{0^-}^\infty dA(x) = 0.$

Then

$$\frac{A^{\dagger}(\theta)S^{\dagger}(\theta)}{-i\theta} \in B^{\dagger}(M;\ 1).$$

Corollary 5.6 is an extension in one direction of Lemma 2.3, although the smoothing magic in that special situation applied when $M(x)$ was any right moment function in the larger class M. If $M(x) = \exp(\sqrt{x}\,)$, for example, then $M(x) \in M$, but

$$xM(x) > \int_0^x M(u)\,du,$$

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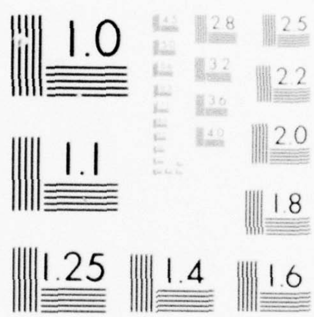

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so that the conclusion of Corollary 5.6 does not necessarily follow from Theorem 5.5. This drawback may be due to an analytic debility in the proof of Theorem 5.5. We suspect, however, that in dealing with situations involving smoothing magic which are more general than that of Lemma 2.3, one can not hope to do better than the class M^*.

5.3 The Variance of the Number of Renewals

In what follows we write $m_r(t) = EN_t^r$ for the rth moment of N_t and $k_r(t)$ for the corresponding rth cumulants. Smith (1959) introduced certain unconventional moments and cumulant which are advantageous for studying cumulants of the number of renewals. The *factorial moments* $\phi_k(t)$, defined for $k \geq 1$ as

$$\phi_k(t) = E\{(N_t+1)(N_t+2) \ldots (N_t+k)\},$$

are, in fact, the coefficients of ζ^k in the expansion for the generating function

$$\Phi_t(\zeta) = E\left\{\frac{1}{(1-\zeta)^{N_t+1}}\right\} .$$

It is possible to write the factorial moments in terms of the conventional moments by using Stirling's numbers of the first kind, and, conversely, the conventional moments can be expressed in terms of the factorial moments by using Stirling's numbers of the second kind:

$$\phi_n(t) = |S_{n+1}^{n+1}|m_n(t) + |S_{n+1}^{n}|m_{n-1}(t) + \ldots + |S_{n+1}^{1}|$$

and

$$m_n(t) = C_{n+1}^{n+1}\,\phi_n(t) - C_{n+1}^{n}\,\phi_{n-1}(t) + \ldots + (-1)^n\,C_{n+1}^{1} .$$

We shall require only the simplest cases of these identities.

Smith (1959) adopted the use of factorial moments, because the Laplace-Stieltjes transform of $\phi_n(t)$ has a particularly convenient form. Throughout our discussion we have employed Fourier-Stieltjes transforms, bearing in mind the possibility of extending our results to unrestricted random variables. Unfortunately $\phi_n(t)$ (for $n \geq 1$) is not a function of bounded variation, so that the Fourier-Stieltjes transform $\phi_n^{\ddagger}(\theta)$ does not exist. To avoid this difficulty we introduce the *modified* factorial moment function

$$\phi_n(t;\zeta,a) = \sum_{k=0}^{\infty}(k+1)(k+2)\dots(k+n)\left\{\zeta^k\int_{-\infty}^{\infty}\Delta_a(x-z)\,dF_k(z)-\zeta^{k+1}\int_{-\infty}^{\infty}\Delta_a(x-z)\,dF_{k+1}(z)\right\},$$

where $0 < \zeta < 1$ and $\Delta_a(x)$ is the triangular density function defined in Section 2.2.

For fixed n, ζ, and $a > 0$, $\phi_n(t;\zeta,a)$ is bounded, nondecreasing, and absolutely continuous. The following lemma (analogous to Lemma 6 of Smith (1959)) shows that the Fourier transform of $\phi_n(t;\zeta,a)$ has a familiar structure:

<u>LEMMA 5.6</u> $$\phi_n^{\dagger}(\theta;\zeta,a) = \frac{n!\,\Delta_a^{\dagger}(\theta)}{\{1-\zeta F^{\ddagger}(\theta)\}^n}\ .$$

<u>PROOF</u>. Transforming term by term we have

$$\phi_n^{\dagger}(\theta;\zeta,a) = \sum_{k=0}^{\infty}(k+1)\ \dots\ (k+n)\left\{\zeta^k\Delta_a^{\dagger}(\theta)[F^{\ddagger}(\theta)]^k - \zeta^{k+1}\Delta_a^{\dagger}(\theta)[F^{\ddagger}(\theta)]^{k+1}\right\}$$

$$= n!\,\Delta_a^{\dagger}(\theta)\left\{1-\zeta F^{\ddagger}(\theta)\right\}\sum_{k=0}^{\infty}\frac{(k+1)\dots(k+n)}{n!}[\zeta F^{\ddagger}(\theta)]^k\ .$$

By applying Newton's formula to sum the series expansion, the lemma follows immediately.

In order to determine Var N_t we shall first find $m_2(t)$ and then apply Theorem 2.4. Lemma 5.6 suggests that the transform approach of Section 2.2 can be extended to prove the following result:

<u>THEOREM 5.7</u> *Let* $F(x) \in \mathcal{D}(M, 3)_\cap C$ *for* $M(x) \in M^*$. *Then* $\phi_2(t) =$ $E\{(N_t+1)(N_t+2)\}$

$$= U(t)\left[\frac{t^2}{\mu_1^2} + \frac{2\mu_2 t}{\mu_1^3}\right] - \frac{2\mu_3}{3\mu_1^3} F_{(3)}(t) + \frac{3\mu_2^2}{2\mu_1^4} F_{(2)}(t) * F_{(2)}(t) + K(t),$$

where $K(t) \in \mathcal{B}(M; 2)$, $K(t)$ *vanishes for* $t < 0$, $\frac{1}{2} K^{\ddagger}(\theta)$ *is given by the right-hand side of* (5.3.3) *below, and*

$$K(t) = o\left(\frac{1}{t^2 M(t)}\right) \quad as \quad t \to \infty.$$

<u>PROOF</u>. Setting $n = 2$ in Lemma 5.6 yields

$$\frac{1}{2}\,\phi_2^{\dagger}(\theta;\zeta,a) = \frac{\Delta_a^{\dagger}(\theta)}{[1-\zeta F^{\ddagger}(\theta)]^2}\,.$$

If we write $\beta = (1-\zeta) - \zeta\mu_1 i\theta$ and apply the expansion

$$F^{\ddagger}(\theta) = 1 + \mu_1 i\theta + \frac{\mu_2}{2}(i\theta)^2 + \frac{\mu_3}{6}(i\theta)^3\ F^{\ddagger}_{(3)}(\theta)$$

given in Theorem 2.1, then

$$(5.3.1) \qquad \frac{1}{2}\,\phi_2^{\dagger}(\theta;\,\zeta,\,a) = \frac{\Delta_{\mathbf{a}}^{\dagger}(\theta)}{[\beta - \zeta\,\frac{\mu_2}{2}\,(i\theta)^2 - \zeta\,\frac{\mu_3}{6}\,(i\theta)^3 F_{(3)}^{\dagger}(\theta)]^2}\ .$$

Let I be a small open interval containing the origin. For $\theta \in I$,

$$\left|\frac{\theta}{\beta}\right|^2 = \frac{\theta^2}{(1-\zeta)^2 + \zeta^2\mu_1^2\theta^2}\ ,$$

and it follows that $|\theta/\beta|^2 \le (1/\mu_1^2) + \theta^2$ uniformly for $0 < \zeta < 1$. Consequently for $\theta \in I$, $|\theta| = O(|\beta|)$ *uniformly with respect to* ζ.

By expanding the right-hand side of (5.3.1) and using the fact that $|\theta| = O(|\beta|)$ we obtain

$$\frac{1}{2}\,\phi_2^{\dagger}(\theta;\,\zeta,\,a) = \frac{\Delta_{\mathbf{a}}^{\dagger}(\theta)}{\beta^2}\left|1 - \frac{\zeta\mu_2(i\theta)^2}{2\beta} - O(|\theta|^2)\right|^{-2}$$

$$= \frac{\Delta_{\mathbf{a}}^{\dagger}(\theta)}{\beta^2}\left\{1 + \frac{\zeta\mu_2(i\theta)^2}{\beta} + O(|\theta|^2)\right\}$$

$$= \Delta_{\mathbf{a}}^{\dagger}(\theta)\left\{\frac{1}{\beta^2} + \frac{(1-\zeta)^2\mu_2}{\zeta\mu_1^2\,\beta^3} - \frac{2(1-\zeta)\mu_2}{\zeta\mu_1^2\,\beta^2} + \frac{\mu_2}{\zeta\mu_1^2\,\beta}\right\} + O(1)\,,$$

since $i\theta = [(1-\zeta) - \beta]/\zeta\mu_1$.

For $\lambda > 0$ and $n > 0$ let

$$e_n(x;\lambda) = \frac{e^{-\lambda x}\,x^{n-1}}{\Gamma(n)}\ , \quad x \ge 0$$

$$= 0, \quad \text{otherwise.}$$

(See the proof for Theorem 5 of Smith (1967; page 294).) The corresponding Fourier transforms are $e_n^{\dagger}(\theta;\lambda) = 1/(\lambda - i\theta)^n$. If we set $\lambda = (1-\zeta)/(\zeta\mu_1)$, then $\zeta^j\mu_1^j/\beta^j$ is the transform $e_j^{\dagger}(\theta;\lambda)$.

Write

$$E(t;\zeta) = \frac{\mu_2}{\zeta^2\mu_1^3}e_1(t;\lambda) + \left[\frac{1}{\zeta^2\mu_1^2} - \frac{2(1-\zeta)\mu_2}{\zeta^3\mu_1^4}\right]e_2(t;\lambda)$$

$$+ \frac{(1-\zeta)^2\mu_2}{\zeta^4\mu_1^5}e_3(t;\lambda).$$

Clearly if $\theta \in I$,

$$\left|\frac{1}{2}\phi_2^{\dagger}(\theta;\zeta,a) - \Delta_a^{\dagger}(\theta)E^{\dagger}(\theta;\zeta)\right| = O(1) \tag{5.3.2}$$

uniformly for $0 < \zeta < 1$.

For $\theta \notin I$ the difference (5.3.2) is bounded as $\zeta \uparrow 1$, since the assumption $F^{\dagger}(\theta) \in C^{\dagger}$ implies that $\sup_{\theta\notin I}|F^{\dagger}(\theta)| < 1$. Both $\frac{1}{2}\phi_2^{\dagger}(\theta;\zeta,a)$ and $\Delta_a^{\dagger}(\theta)\,E^{\dagger}(\theta;\zeta)$ are integrable for $\theta \notin I$, since $\Delta_a^{\dagger}(\theta)$ is integrable. Therefore we may write

$$\frac{1}{2}\phi_2(t;\zeta,a) - \Delta_a(t)*E(t;\zeta) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\left\{\frac{1}{2}\phi_2^{\dagger}(\theta;\zeta,a) - \Delta_a^{\dagger}(\theta)E^{\dagger}(\theta;\zeta)\right\}d\theta.$$

Setting $\phi_2(t;1,a) = \lim_{\zeta\uparrow 1}\phi_2(t;\zeta,a)$ and $E(t;1) = \lim_{\zeta\uparrow 1}E(t;\zeta)$, it follows by bounded convergence that

$$\frac{1}{2}\,\phi_2(t;\,1,\,a) - \Delta_a(t)*E(t;\,1) =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\,\Delta_a^{\dagger}(\theta)\left\{\frac{1}{[1-F^{\dagger}(\theta)]^2} - \frac{1}{(-\mu_1 i\theta)^2} - \frac{\mu_2}{\mu_1^3(-i\theta)}\right\}d\theta\,.$$

A *formal* expansion of $1/[1 - F^{\dagger}(\theta)]^2$ yields

$$\frac{1}{[1-F^{\dagger}(\theta)]^2} = \frac{1}{(-\mu_1 i\theta)^2}\;\frac{1}{[1-(1-F^{\dagger}_{(1)}(\theta))]^2}$$

$$= \frac{1}{(-\mu_1 i\theta)^2}\left\{1 + 2(1-F^{\dagger}_{(1)}(\theta)) + \frac{(1-F^{\dagger}_{(1)}(\theta))^2\{3-2(1-F^{\dagger}_{(1)}(\theta))\}}{[1-(1-F^{\dagger}_{(1)}(\theta))]^2}\right.$$

$$= \frac{1}{(-\mu_1 i\theta)^2} + \frac{\mu_2}{\mu_1^3(-i\theta)} - \frac{\mu_3}{3\mu_1^3}\,F^{\dagger}_{(3)}(\theta) + \frac{3\mu_2^2}{4\mu_1^4}\left|\,F^{\dagger}_{(2)}(\theta)\right|^2 + K^{\dagger}(\theta)\,,$$

where

$$K^{\dagger}(\theta) = \frac{[1-F^{\dagger}_{(1)}(\theta)]^2}{-\mu_1 i\theta[F^{\dagger}_{(1)}(\theta)]^2}\left\{\frac{1+2F^{\dagger}_{(1)}(\theta)-3[F^{\dagger}_{(1)}(\theta)]^2}{-\mu_1 i\theta}\right\}. \tag{5.3.3}$$

Note that we have used the facts (see Theorem 2.1) that

$$F^{\dagger}_{(2)}(\theta) = \frac{1-F^{\dagger}_{(1)}(\theta)}{\frac{-\mu_2}{2\mu_1}\,i\theta}$$

and

$$F^{\dagger}_{(3)}(\theta) = \frac{1-F^{\dagger}_{(2)}(\theta)}{\frac{-\mu_3}{3\mu_2}\,i\theta}\,.$$

As in Section 2.2 write $q^{\dagger}(\theta)$ for the special SMF $q^{\dagger}(\theta; -2, -1, 1, 2)$ and define

$$K_1^{\ddagger}(\theta) = q^{\dagger}(\theta)M^{\ddagger}(\theta)$$

$$K_2^{\ddagger}(\theta) = [1-q^{\dagger}(\theta)]K^{\ddagger}(\theta) .$$

Clearly we may write $K_1^{\ddagger}(\theta) = K_{11}^{\ddagger}(\theta)K_{12}^{\ddagger}(\theta)K_{13}^{\ddagger}(\theta)$, where

$$K_{11}^{\ddagger}(\theta) = \frac{[1-F_{(1)}^{\ddagger}(\theta)]^2}{-\mu_1 i\theta} ,$$

$$K_{12}^{\ddagger}(\theta) = 1 + 2F_{(1)}^{\ddagger}(\theta) - 3[F_{(1)}^{\ddagger}(\theta)]^2,$$

and

$$K_{13}^{\ddagger}(\theta) = \frac{q^{\dagger}(\theta)}{[F_{(1)}^{\ddagger}(\theta)]^2} .$$

Since $M(x) \in M^*$, it follows that $M^\circ(x) = xM(x) \in M^*$ and consequently $F(x) \in \mathcal{D}(M^\circ; 2) \cap C$. By the smoothing magic of Lemma 2.3, $K_{11}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M^\circ;1)$ or, equivalently, $K_{11}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$. Note that $K_{12}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$, so that *both* $K_{11}^{\ddagger}(\theta)$ and $K_{12}^{\ddagger}(\theta)$ belong to the class $\mathcal{B}^{\ddagger}(M^\circ; 1)$. $K_{11}(x)$ is absolutely continuous to the right of the origin and vanishes at infinity; furthermore

$$\lim_{\theta\to 0} K_{12}^{\ddagger}(\theta) = 0.$$

Therefore we can apply the smoothing magic of Corollary 5.6 to conclude that $K_{11}^{\ddagger}(\theta)\, K_{12}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M^\circ; 1)$ or, equivalently $K_{11}^{\ddagger}(\theta)\, K_{12}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$. Setting $J = [-2,2]$, $z = F_{(1)}^{\ddagger}(\theta)$, $\Phi(z) = 1/z^2$, and $\psi(\theta) = q^{\dagger}(\theta)$, it follows by Part A of Theorem 2.2 that $K_{13}^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$. (See the proof of Theorem 2.4.) Therefore $K_1^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$.

It is not so difficult to handle the term

$$K_2^{\ddagger}(\theta) = \frac{[1-q^{\dagger}(\theta)][1-F^{\ddagger}_{(1)}(\theta)]^2\{1+2F^{\ddagger}_{(1)}(\theta)-3[F^{\ddagger}_{(1)}(\theta)]^2\}}{[1 - F^{\ddagger}(\theta)]^2} .$$

Write $J = (-\infty, 2) \cup (2, \infty)$, $z = F^{\ddagger}(\theta)$, $\Phi(z) = 1/(1-z)^2$, and $\psi(\theta) = 1 - q^{\dagger}(\theta)$. Then by Part B of Theorem 2.2,

$$\frac{1 - q^{\dagger}(\theta)}{[1-F^{\ddagger}(\theta)]^2} \in \mathcal{B}^{\ddagger}(M; 3),$$

so that $K_2^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2)$. Consequently

$$K^{\ddagger}(\theta) = K_1^{\ddagger}(\theta) + K_2^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M; 2). \tag{5.3.4}$$

We remark at this point that Theorem 5.5 can easily be reproved using the moment class $\mathcal{B}(M_I; 1)$ in place of $\mathcal{B}(M_I; 0)$. Using this result with $M(x)$ in the more general moment class M, it is possible to show a bit more, namely that

$$K^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}\left(\int_0^x M(u)du; 1\right). \tag{5.3.5}$$

Clearly (5.3.4) implies (5.3.5) when $M(x)$ satisfies the additional growth restriction $M(2x) = O(M(x))$.

Since we have shown that

$$\frac{1}{[1-F^{\ddagger}(\theta)]^2} - \frac{1}{(-\mu_1 i\theta)^2} - \frac{\mu_2}{\mu_1^3(-i\theta)}$$

is the Fourier-Stieltjes transform of the function

$$-\frac{\mu_3}{3\mu_1^3} F_{(3)}(t) + \frac{\mu_2^2}{4\mu_1^4} F_{(2)}(t)*F_{(2)}(t) + K(t),$$

it follows that

$$\Delta_a^{\dagger}(\theta)\left\{\frac{1}{[1-F^{\ddagger}(\theta)]^2} - \frac{1}{(-\mu_1 i\theta)^2} - \frac{\mu_2}{\mu_1^3(-i\theta)}\right\}$$

is the Fourier transform of

$$\int_{-\infty}^{\infty} \Delta_a(t-z)d_z\left\{-\frac{\mu_3}{3\mu_1^3} F_{(3)}(z) + \frac{3\mu_2^2}{4\mu_1^4} F_{(2)}(z)*F_{(2)}(z) + K(z)\right\} .$$

Therefore

$$\frac{1}{2}\phi_2(t, 1, a) - \int_{-\infty}^{\infty} \Delta_a(t-z)d_z\left\{-\frac{\mu_3}{3\mu_1^3} F_{(3)}(z) + \right.$$

$$(5.3.6) \qquad \left. + \frac{3\mu_2^2}{4\mu_1^4} F_{(2)}(z)*F_{(2)}(z) + K(z)\right\} = \Delta_a(t)*E(t; 1).$$

Rewriting (5.3.6) we obtain

$$\frac{1}{2}\int_{-\infty}^{\infty} \Delta_a(t-z)d\phi_2(z) = \int_{-\infty}^{\infty} \Delta_a(t-z)d_z\left\{\int_{-\infty}^{z} E(u; 1)du\right.$$

$$\left. - \frac{\mu_3}{3\mu_1^3} F_{(3)}(z) + \frac{3\mu_2^2}{4\mu_1^4} F_{(2)}(z)*F_{(2)}(z) + K(z)\right\}$$

$$= \int_{-\infty}^{\infty} \Delta_a(t-z)d_z\left\{U(z)\left[-\frac{z^2}{2\mu_1^2} + \frac{\mu_2 z}{\mu_1^3}\right] - \frac{\mu_3}{3\mu_1^3} F_{(3)}(z)\right.$$

$$\left. + \frac{3\mu_2^2}{4\mu_1^4} F_{(2)}(z)*F_{(2)}(z) + K(z)\right\} .$$

As in Section 2.2 we may use a "sandwiching" process to approximate characteristic functions of intervals by linear combinations of triangular densities. Then by a standard extension argument we obtain

$$\phi_2(t) = U(t)\left[-\frac{t^2}{\mu_1^2} + \frac{2\mu_2 t}{\mu_1^3}\right] - \frac{2\mu_3}{3\mu_1^3} F_{(3)}(t)$$

$$+ \frac{3\mu_2^2}{2\mu_1^4} F_{(2)}(t)*F_{(2)}(t) + 2K(t) + C.$$

We conclude the proof of Theorem 5.7 by noting that $\phi_2(t) \to 0$ as $t \to -\infty$ by the Strong Law of Large Numbers; consequently $C = -K(-\infty) = 0$.

We now apply Theorems 2.4 and 5.7 to obtain the following improvement of Corollary 5.3:

<u>THEOREM 5.8</u> *If* $F(x) \in \mathcal{D}(M; 3)_{\cap} C$ *for some* $M(x) \in M^*$, *then as* $t \to \infty$,

$$\text{Var } N_t = \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2}\right)$$

$$+ \frac{\mu_2 t}{\mu_1^3} [1 - F_{(2)}(t)] + \frac{2\mu_3}{3\mu_1^3} [1 - F_{(3)}(t)] + o\left(\frac{1}{tM(t)}\right).$$

<u>PROOF</u>. The argument is entirely straightforward, although computationally detailed. Let $t > 0$; then by Theorem 2.4

$$\{EN_t\}^2 = \frac{t^2}{\mu_1^2} - \frac{2t}{\mu_1} + 1 + \left[\frac{\mu_2 t}{\mu_1^3} - \frac{\mu_2}{\mu_1^2}\right] F_{(2)}(t)$$

$$+ \frac{\mu_2^2}{4\mu_1^4} F_{(2)}^2(t) + \frac{\mu_2}{\mu_1^2} F_{(2)}(t)L(t) + \frac{2t}{\mu_1} L(t)$$

$$+ L^2(t) - 2L(t),$$

where the remainder function $L(t)$ is now in the class $\mathcal{B}(M; 2)$. On the other hand, by Theorems 2.4 and 5.7

$$E\{N_t^2\} = \frac{t^2}{\mu_1^2} + \left[\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}\right] t + 1 - \frac{3\mu_2}{2\mu_1^2} F_{(2)}(t)$$

$$(5.3.7) \qquad + \frac{3\mu_2^2}{2\mu_1^4} F_{(2)}(t)*F_{(2)}(t) - \frac{2\mu_3}{3\mu_1^3} F_{(3)}(t)$$

$$+ K(t) - 3L(t),$$

where the remainder function $K(t)$ is in the class $\mathcal{B}(M; 2)$. Consequently

$$\mathrm{Var}\, N_t = E\{N_t^2\} - \{EN_t\}^2 =$$

$$= \frac{\mu_2 - \mu_1^2}{\mu_1^3} t + \left(\frac{5\mu_2^2}{4\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{\mu_2}{2\mu_1^2}\right) + \frac{2\mu_3}{3\mu_1^3} [1 - F_{(3)}(t)]$$

$$+ \frac{\mu_2 t}{\mu_1^3} [1 - F_{(2)}(t)] + R(t),$$

where $R(t) = \frac{\mu_2}{2\mu_1^2} [1 - F_{(2)}(t)] + \frac{\mu_2^2}{4\mu_1^4} [1 - F_{(2)}^2(t)]$

$$- \frac{3\mu_2^2}{2\mu_1^4} [1 - F_{(2)}(t)*F_{(2)}(t)] - L(t) - L^2(t) + K(t)$$

$$- \frac{2t}{\mu_1} L(t) - \frac{\mu_2}{\mu_1^2} F_{(2)}(t) L(t).$$

Since $F_{(2)}(t) \in \mathcal{D}(M; 1)$,

$$tM(t)[1 - F_{(2)}(t)] \le \int_t^{\infty} uM(u)\,dF_{(2)}(u) \to 0$$

as $t \to \infty$, so that $[1 - F_{(2)}(t)] = o(1/tM(t))$. It can similarly be shown that every term in the expression for $R(t)$ is $o(1/tM(t))$ as $t \to \infty$. On the other hand, both

$$\frac{2\mu_3}{3\mu_1^3}[1 - F_{(3)}(t)] \quad \text{and} \quad \frac{\mu_2 t}{\mu_1^3}[1 - F_{(2)}(t)]$$

are of magnitude $o(1/tM(t))$ as $t \to \infty$.

An immediate application of Theorem 5.8 is the following:

<u>COROLLARY 5.9</u> *If* $V_N(t)$ *is the variance of the number of events occurring by time* t *in a superposition of* N *iid renewal processes with lifetime distribution* $F(x) \in \mathcal{D}(M;\,3) \cap C$ *for* $M(x) \in M^*$, *then*

$$V_N(t) = \frac{N\sigma^2 t}{\mu_1^3} + N\left(\frac{1}{6} + \frac{\sigma^4}{2\mu_1^4} - \frac{\mu_3}{3\mu_1^3}\right) +$$

$$+ \frac{N\mu_2 t}{\mu_1^3}[1 - F_{(2)}(t)] + \frac{2N\mu_3}{3\mu_1^3}[1 - F_{(3)}(t)]$$

$$+ o\left(\frac{1}{tM(t)}\right) \quad \text{as} \quad t \to \infty,$$

where μ_i *denotes the* ith *moment of* $F(x)$, *and* $\sigma^2 = \mu_2 - \mu_1^2$.

Corollary 5.9 could conceivably be used develop an improved variance-time curve approach for statistical analysis of superposition.

5.4 The Third Cumulant of the Number of Renewals

The methods used to prove Theorems 5.7 and 5.8 can be employed to obtain detailed expansion for the nth cumulant (semi-invariant) of N_t. For $n = 3$ we can show the following:

THEOREM 5.10 *If* $F(x) \in \mathcal{D}(M; 4) \cap C$ *for some* $M(x) \in M^*$, *then as* $t \to \infty$,

$$k_3(t) = E\{N_t - EN_t\}^3 =$$

$$= \left[\frac{1}{\mu_1} - \frac{6\mu_2}{\mu_1^3} - \frac{\mu_3}{\mu_1^4} + \frac{9\mu_2^2}{\mu_1^5}\right]t + \frac{3\mu_4}{4\mu_1^4}$$

$$- \frac{3\mu_2 t^2}{2\mu_1^4}[1 - F_{(2)}(t)] - \frac{2\mu_3 t}{\mu_1^4}[1 - F_{(3)}(t)]$$

$$- \frac{3\mu_4}{4\mu_1^4}[1 - F_{(4)}(t)] + o\left(\frac{1}{tM(t)}\right) .$$

OUTLINE OF PROOF. Since the proof is analogous to that of Theorem 5.8, we shall describe only the major steps involved. The third cumulant of N_t can be written as

$$k_3(t) = EN_t^3 - 3EN_tEN_t^2 + 2\{EN_t\}^3 .$$

We have already developed expressions for EN_t and EN_t^2. To obtain an expansion for $m_3(t) = EN_t^3$ we need a result similar to Theorem 5.7 for the factorial moment

$$\phi_3(t) = E\left\{(N_t+1)(N_t+2)(N_t+3)\right\} .$$

Although $\phi_3(t)$ does not possess a Fourier-Stieltjes transform, a formal expansion and application of Theorem 2.1 yield

$$\frac{3!}{\{1-F^{\ddagger}(\theta)\}^3} = \left[\frac{9\mu_2^2}{\mu_1^5} - \frac{3\mu_3}{\mu_1^4}\right]\frac{1}{-i\theta} + \frac{9\mu_2}{\mu_1^4(-i\theta)^2} + \frac{6}{\mu_1^3(-i\theta)}$$

$$+ \frac{3\mu_4}{4\mu_1^4} F^{\ddagger}_{(4)}(\theta) - \frac{3\mu_2\mu_3}{\mu_1^5} F^{\ddagger}_{(2)}(\theta)F^{\ddagger}_{(3)}(\theta)$$

$$- \frac{3\mu_2\mu_3}{\mu_1^5} F^{\ddagger}_{(3)}(\theta) + \frac{15\mu_2^3}{2\mu_1^6}[F^{\ddagger}_{(1)}(\theta)]^3 + C^{\ddagger}(\theta),$$

where

$$C^{\ddagger}(\theta) = \frac{6[1-F^{\ddagger}_{(1)}(\theta)]^3\{10-15(1-F^{\ddagger}_{(1)}(\theta)) + 6(1-F^{\ddagger}_{(1)}(\theta))^2\}}{[-\mu_1 i\theta\ F^{\ddagger}_{(1)}(\theta)]^3}.$$

The smoothing magic of Lemma 2.3 and Theorem 5.5, together with Theorem 2.2 (Wiener-Pitt-Lévy-Smith), can be used to show that $C^{\ddagger}(\theta) \in \mathcal{B}^{\ddagger}(M;\ 3)$. By introducing a modified factorial moment function $\phi_3(t;\ \zeta,\ a)$ and following the lines of the proof of Theorem 5.7, we obtain

$$\phi_3(t) = U(t)\left[\frac{t^3}{\mu_1^3} + \frac{9\mu_2 t^2}{2\mu_1^4} + \left(\frac{9\mu_2^2}{\mu_1^3} - \frac{3\mu_3}{\mu_1^4}\right)t\right]$$

$$+ \frac{15\mu_2^3}{2\mu_1^6} F_{(1)}{*}F_{(1)}{*}F_{(1)}(t) - \frac{3\mu_2\mu_3}{\mu_1^5} F_{(2)}(t){*}F_{(3)}(t)$$

$$- \frac{3\mu_2\mu_3}{\mu_1^5} F_{(3)}(t) + \frac{3\mu_4}{4\mu_1^4} F_{(4)}(t) + C(t),$$

where $C(t) \in \mathcal{B}(M;\ 3)$, $C(t)$ vanishes for $t < 0$, and

$$C(t) = o\left(\frac{1}{t^3 M(t)}\right) \quad \text{as} \quad t \to \infty.$$

Theorem 2.4 and (5.3.7) in the proof of Theorem 5.8 can be applied to develop an expansion for $m_3(t)$. The desired result follows after a considerable amount of computation and simplification.

Using the methods of Theorems 5.8 and 5.10 we can find expansions for $k_n(t)$ when $n > 3$, although the computation involved is formidable. In general we need to assume that $F(x) \in \mathcal{D}(M; n+1) \cap C$ for some $M(x) \in \mathcal{M}^*$; then there exist constants a_n, b_n, c_{n2}, c_{n3}, ..., and $c_{n,n+1}$ such that

$$(5.3.8) \qquad k_n(t) = a_n t + b_n + \sum_{j=2}^{n+1} c_{nj} t^{n+1-j}[1 - F_{(j)}(t)] + o\left(\frac{1}{tM(t)}\right)$$

as $t \to \infty$. Although (5.3.8) suggests a sharpening of Theorem 5.2, both in terms of the generalized moment assumption and the more detailed remainder term, the constants involved are, nevertheless, difficult to evaluate. Further work (most likely requiring the use of deeper combinatorial methods) is needed to establish (5.3.8) rigorously and to provide usable algorithms for computing the constants.

CHAPTER VI: MOMENTS OF THE FORWARD RECURRENCE-TIME

Given a sequence of renewal lifetimes $\{X_n\}_{n=1}^{\infty}$ the *forward recurrence-time* ζ_t is defined as the time measured forward from t to the next renewal. In other words, ζ_t is the residual lifetime of the component in use at time t. We shall write $G_t(x) = P\{\zeta_t \le x\}$ for the distribution function of ζ_t. Using a familiar renewal argument it can be shown that

$$(6.0.1) \qquad P\{\zeta_t > x\} = 1 - G_t(x) = 1 - F(t+x) + \int_o^t \{1-F(t+x-u)\}dH(u)\,,$$

where $F(x) = P\{X_n \le x\}$ and $H(t)$ is the renewal function. By applying the Key Renewal Theorem to the right-hand side of (6.0.1), it follows that

$$(6.0.2) \qquad \lim_{t\to\infty} G_t(x) = \int_o^x \frac{1-F(u)}{\mu_1}\, du = F_{(1)}(x)\,,$$

provided $F(x) \in \mathcal{D}(I;\, 1)$.

Using Wald's Identity and the Second Renewal Theorem one can prove that

$$(6.0.3) \qquad E\zeta_t = \frac{\mu_2}{2\mu_1} + o(1) \quad \text{as} \quad t \to \infty,$$

assuming $F(x) \in \mathcal{D}(I;\, 2)$. Apparently this approach does not yield similar expressions for higher moments of ζ_t.

In Section 6.1 we shall describe a method for obtaining refined estimates of the moments of ζ_t. An application of the expressions derived for $E\zeta_t$ and $E[\zeta_t^2]$ will be given in Section 6.2, which deals with the variance of the number of renewals in an interval away from the origin. This result, in turn, can be used to find the covariance of the numbers of renewals in disjoint intervals.

6.1 Estimates for $E\zeta_t$ and $E[\zeta_t^2]$

Define, as usual, $S_o = 0$ and $S_n = X_1 + \ldots + X_n$ for $n \geq 1$. Since the "overshoot" ζ_t can be written as

$$\zeta_t = S_{N_t+1} - t,$$

it follows that

$$(6.1.1) \qquad \zeta_t + t = \sum_{i=1}^{\infty} X_i \, U(t - S_{i-1}),$$

where $U(x) = P\{0 \leq x\}$ is the unit function. (6.1.1) is a version of a more general identity which has been used in the study of cumulative renewal processes. By taking the expectation of both sides of (6.1.1) and using the independence of X_i and S_{i-1}, we obtain

$$E\zeta_t + t = \mu_1\{1 + H(t)\}.$$

Assuming that $F(x) \in \mathcal{D}(M; 2)_{\cap} C$ for some $M(x) \in M$, an application of Theorem 2.4 yields the following refinement of (6.0.3):

$$E\zeta_t = \frac{\mu_2}{2\mu_1} - \frac{\mu_2}{2\mu_1}[1 - F_{(2)}(t)] + L(t),$$

where $L(t) = o(1/tM(t))$ as $t \to \infty$.

Using (6.1.1) as a starting point, we can develop similar expressions for higher moments of ζ_t, although the calculations involved are lengthy. We now derive an estimate for $E[\zeta_t^2]$ to illustrate the approach.

By squaring both sides of (6.1.1), it follows that

$$(\zeta_t+t)^2 = \sum_{i=1}^{\infty} X_i^2 U(t-S_{i-1}) + 2 \sum_{r>s\geq 1}\sum X_r X_x U(t-S_{r-1}),$$

since $U(t-S_{r-1})U(t-S_{s-1}) = U(t-S_{r-1})$ if $r > s$. This implies that

$$(6.1.2) \qquad E\{(\zeta_t+t)^2\} = \mu_2\{1+H(t)\} + 2\mu_1 \sum_{r>s\geq 1}\sum E\{X_s U(t-S_{r-1})\} .$$

Using $F^{*n}(t)$ to denote the nth iterated convolution of $F(t)$ with itself,

$$E\{X_s U(t-S_{r-1})|X_s\} = X_s P\{S_{r-1} \leq t|X_s\}$$

$$= X_s \cdot F_{X_s}(t) * F^{*(r-2)}(t),$$

where $F_{X_s}(t) = I_{(X_s,\infty)}(t)$. Therefore

$$(6.1.3) \qquad E\{X_s U(t-S_{r-1})|X_s\} = \int_0^t X_s\, I_{(X_s,\infty)}(t-u)dF^{*(r-2)}(u) .$$

By taking expectations on both sides of (6.1.3) and applying Fubini's theorem, we obtain

$$E\{X_s U(t-S_{r-1})\} = \int_0^t \int_0^{\infty} x I_{(x,\infty)}(t-u)dF(x)dF^{*(r-2)}(u)$$

$$= \int_0^t \int_0^{t-u} x dF(x)dF^{*(r-2)}(u) = R(t)*F^{*(r-2)}(u),$$

where $R(t) = \int_0^t u dF(u)$.

Clearly $\mu_1^{-1}R(t)$ is the distribution function of a positive random variable. (It is interesting to note that, in particular, $\mu_1^{-1}R(x)$ arises as the limiting distribution as $t \to \infty$ of the length of the renewal lifetime containing t or, in other words, the total life of the component in current operation ; however, in the present context $\mu_1^{-1}R(x)$ does not appear to have such an interpretation.) For $k = 1,2,\ldots$,

$$\int_0^\infty t^k dR(t) = \int_0^\infty t^{k+1}\, dF(t) = \mu_{k+1},$$

i.e., the kth moment of $\mu_1^{-1}R(t)$ exists, provided $F(t) \in \mathcal{D}(I;\, k)$, and is equal to μ_{k+1}/μ_1. Consequently Theorem 2.1 can be used to develop expansions for the characteristic function of $\mu_1^{-1}R(t)$, and, furthermore, it can be shown that the Fourier-Stieltjes transform of $R(t)$ is given by

$$R^{\dagger}(\theta) = -iF^{\dagger\prime}(\theta). \tag{6.1.4}$$

Therefore $E\{X_s U(t-S_{r-1})\}$ has Fourier-Stieltjes transform

$$R^{\dagger}(\theta)F^{\dagger}(\theta) = -iF^{\dagger\prime}(\theta)[F^{\dagger}(\theta)]^{r-2}.$$

Since the double sum

$$S(t) = \sum_{r>s\geq 1}\sum E\{X_s U(t-S_{r-1})\}$$

is nondecreasing but unbounded as $t \to \infty$, it does not possess a legitimate Fourier-Stieltjes transform. This difficulty can be overcome by introducing a smoothed bounded modification of $S(t)$. We have already employed such a procedure in Sections 2.2 and 5.3, and rather than duplicate those arguments here, we shall simply use transforms defined in a formal sense. (N.B. Since we are dealing entirely with positive random variables, this technicality could be conveniently eliminated by working with Laplace-

Stieltjes transforms; however this would preclude possible future extensions to unrestricted random variables.)

The *formal* Fourier-Stieltjes transform of $S(t)$ is given by

$$S^{\dagger}(\theta) = \frac{iF^{\dagger\prime}(\theta)}{[1-F^{\dagger}(\theta)]^2} = -i\frac{d}{d\theta}\left[\frac{1}{1-F^{\dagger}(\theta)}\right].$$

Formally $1/[1-F^{\dagger}(\theta)]$ corresponds to the renewal function $H_{(t)} + U(t)$, and we recognize "$-i\, d/d\theta$" as the "first moment" operator.

Assuming that $F(x) \in \mathcal{D}(M; 3)_{\cap} C$ for some $M(x) \in M$, it follows from Theorem 2.2 that

$$S(t) = \int_0^t u d[H(t) + U(t)\}$$

$$= \frac{t^2}{2\mu_1} + \frac{\mu_2}{2\mu_1^2}\int_0^t u dF_{(2)}(u) + \int_0^t u dL(u).$$

Note that $\int_0^t u dL(u)$ belongs to the class $B(M; 1)$ and corresponds to $-iL^{\dagger\prime}(\theta)$, where

(6.1.5) $$L^{\dagger}(\theta) = \frac{[1 - F^{\dagger}_{(1)}(\theta)]^2}{1 - F^{\dagger}(\theta)}.$$

Differentiating (6.1.5) with respect to θ and using the facts (see Theorem 2.1) that

$$1 - F^{\dagger}(\theta) = -\mu_1 i\theta F^{*}_{(1)}(\theta) \quad \text{and} \quad 1 - F^{\dagger}_{(1)}(\theta) = \frac{-\mu_2}{2\mu_1} i\theta F^{\dagger}_{(2)}(\theta),$$

we can show that

$$\lim_{\theta\to 0} L^{\dagger\prime}(\theta) = \frac{-\mu_2^2}{4\mu_1^3}.$$

Therefore

$$L(t) + \frac{\mu_2^2}{4\mu_1^3} = o\left[\frac{1}{tM(t)}\right] \quad \text{as} \quad t \to \infty.$$

By expanding the left-hand side of (6.1.2), substituting, and collecting terms, we obtain

$$E[\zeta_t^2] = \frac{\mu_2 t}{\mu_1}\,[1 - F_{(2)}(t)] - \frac{\mu_2^2}{2\mu_1^2}\,[1 - F_{(2)}(t)] \tag{6.1.6}$$

$$+ \frac{\mu_2}{\mu_1}\int_0^t u\,dF_{(2)}(u) + o\left[\frac{1}{tM(t)}\right] \quad \text{as} \quad t \to \infty.$$

For large t the first term on the right-hand side of (6.1.6) is $o(1/M(t))$; the second term is $o(1/tM(t))$. Since the first moment of the second derived distribution is $\mu_3/3\mu_2$,

$$\lim_{t\to\infty} E[\zeta_t^2] = \frac{\mu_3}{3\mu_1}.$$

This is not surprising in view of (6.0.2), since the second moment of the first derived distribution is $\mu_3/3\mu_1$.

The following result summarizes the preceding discussion:

<u>THEOREM 6.1</u> *Suppose* $F(x) \in \mathcal{D}(M;\, 2)_{\cap}\, C$ *for some* $M(x) \in \mathcal{M}$. *Then*

$$E[\zeta_t] = \frac{\mu_2}{2\mu_1}\,F_{(2)}(t) + o\left[\frac{1}{tM(t)}\right] \quad \text{as} \quad t \to \infty.$$

If additionally $F(x) \in \mathcal{D}(M;\, 3)_{\cap}\, C$, *then*

$$E[\zeta_t^2] = \frac{\mu_2 t}{\mu_1}\,[1 - F_{(2)}(t)] + \frac{\mu_2}{\mu_1}\int_0^t u\,dF_{(2)}(u) + o\left[\frac{1}{tM(t)}\right] \quad \text{as} \quad t \to \infty.$$

By raising both sides of (6.1.1) to the nth power and taking their expectations, it should be possible to find an expansion for $E[\zeta_t^n]$ with a remainder term of magnitude $o(1/tM(t))$. Undoubtedly this entails the assumption that $F(x) \in \mathcal{D}(M; n+1) \cap C$. The computation involved for each $n \geq 3$ is formidable, but it may be possible via more extensive combinatorial work to obtain a closed expression for the nth moment of ζ_t.

It can easily be shown using integration by parts that

$$\int_0^\infty x^k dF_{(1)}(x) = \frac{\mu_{k+1}}{(k+1)/\mu_1} \quad .$$

One might therefore suspect that

$$\lim_{t\to\infty} E[\zeta_t^n] = \frac{\mu_{n+1}}{(n+1)/\mu_1} \; ; \tag{6.1.7}$$

in fact, (6.1.7) is a consequence of the following fact which will later prove useful in a different setting:

<u>LEMMA 6.2</u> *Suppose* $F(x) \in \mathcal{D}(M; 1) \cap C$ *for some* $M(x) \in \mathcal{M}$ *and* ζ_t *is the forward recurrence-time corresponding to the renewal lifetimes* $\{X_n\}_{n=1}^\infty$ *with distribution* $F(x)$. *Then there exists an absolutely continuous distribution function* $D(x) \in \mathcal{D}(M; 0)$ *such that* $D(0) = 0$ *and for* $x > 0$,

$$P\{\zeta_t > x\} = 1 - G_t(x) \leq 1 - D(x), \quad (t > 0).$$

<u>PROOF</u>. By the uniformly bounded variation (UBV) property of the renewal function, there exists a constant $A > 0$ such that for all $u > 0$,

$$H(u + 1) - H(u) \le A.$$

(Although the UBV property follows from Theorem 2.4, it can be obtained directly without the aid of such a strong result and has, in fact, been used by authors to prove the elementary renewal theorem.) Let $A' = \max(A, 1, \mu_1^{-1})$; then by (6.0.1),

$$1 - G_t(x) \le 1 - F(t+x) + \sum_{j=1}^{[t]+1} \int_{j-1}^{j} \{1 - F(t+x-u)\} dH(u)$$

$$\le 1 - F(t+x) + \sum_{j=1}^{[t]+1} [1 - F(t+x-j)][H(j) - H(j-1)]$$

$$\le A' \sum_{j=0}^{[t]+1} [1 - F(t+x-j)] \le A' \int_0^{[t]+2} [1 - F(t+x-u)] du$$

$$\le \mu_1 A' [1 - F_{(1)}(x+t-[t]-2)] \le \mu_1 A' [1 - F_{(1)}(x-2)].$$

Define $D(x) = 0$ for $x \le 2$. Since $F_{(1)}(x)$ is absolutely continuous and $\mu_1 A' > 1$, there exists a constant $\Delta > 2$ such that $\mu_1 A'[1 - F_{(1)}(x-2)] = 1$. Take $D(x) = 0$ for $2 < x \le \Delta$, and define

$$1 - D(x) = \mu_1 A'[1 - F_{(1)}(x-2)], \qquad x > \Delta.$$

It is then easy to verify that $D(x)$ has the properties claimed in the statement of the lemma.

Returning to (6.1.7), we can apply Fubini's theorem to write

$$E[\zeta_t^n] = \int_0^\infty \int_0^x nu^{n-1}\, du dG_t(x) = \int_0^\infty nu^{n-1} \int_u^\infty dG_t(x) du$$

$$= \int_0^\infty nu^{n-1} [1 - G_t(u)] du .$$

By Lemma 6.2 and Fubini's theorem

$$\int_0^\infty nu^{n-1}[1 - G_t(u)]du \le \int_0^\infty nu^{n-1}[1 - D(u)]du$$

$$= \int_0^\infty \int_0^x nu^{n-1}\, du dD(x) = \int_0^\infty x^n\, dD(x) < \infty,$$

provided $F(x) \in \mathcal{D}(I;\, n+1)$. Convergence of moments (6.1.7) follows by dominated convergence.

6.2 The Variance of the Number of Renewals in an Interval Away From the Origin

Since the forward recurrence-time ζ_t plays a role in the study of modified renewal processes and occurs in a number of applications (such as counter models), the results of the previous section should be useful in a variety of situations. Here we shall use Theorem 6.1 to derive estimates for the expectation and variance of the number of renewals in an interval away from the origin.

Write $N(t;\, T)$ for the number of renewals occurring in the time interval $(t, t + T]$, $t > 0$ and $T > 0$. We assume that t is fixed (with respect to T) and is moderately large in the sense that ζ_t does not necessarily possess the limit distribution given by (6.0.2). Furthermore we suppose $F(x) \in \mathcal{D}(M;\, 2) \cap C$ for some $M(x) \in \mathcal{M}$. Then by Theorem 2.4

$$E\{[N(t;\, T)-1] \,|\zeta_t\} = \frac{(T-\zeta_t)}{\mu_1} - 1 + \frac{\mu_2}{2\mu_1^2} F_{(2)}(T-\zeta_t) + L(T-\zeta_t), \tag{6.2.1}$$

where $L(x) \in \mathcal{B}(M;\, 1)$. (We subtract one renewal from $N(t;\, T)$, since Theorem 2.4 applies to renewal processes whose first lifetime begins at the origin.)

If, moreover, $F(x) \in \mathcal{D}(M; 3)_{\cap} C$ for some $M(x) \in M^*$, then (5.3.7) in the proof of Theorem 5.8 implies that

$$E\{[N(t;\,T)-1]^2 | \zeta_t\} = \frac{(T-\zeta_t)^2}{\mu_1^2} + \left[\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}\right](T-\zeta_t)$$

$$+ 1 - \frac{3\mu_2}{2\mu_1^2} + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} + \frac{3\mu_2}{2\mu_1^2}[1 - F_{(2)}(T-\zeta_t)]$$

(6.2.2)

$$- \frac{3\mu_2^2}{2\mu_1^4}[1 - F_{(2)}*F_{(2)}(T-\zeta_t)] + \frac{2\mu_3}{3\mu_1^3}[1 - F_{(3)}(T-\zeta_t)]$$

$$+ k(T-\zeta_t) - 3L(T-\zeta_t),$$

where both $L(x)$ and $K(x)$ belong to $\mathcal{B}(M; 2)$.

By taking the expectations of both sides of (6.2.1) and (6.2.2) we obtain the (rather complicated) expressions

$$E\{N(t;\,T)\} = \frac{T}{\mu_1} - \frac{E\zeta_t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} F_{(2)}*G_t(T) + L*G_t(T)$$

and

$$E\{[N(t;\,T)]^2\} = \frac{T^2}{\mu_1^2} - \frac{2TE\zeta_t}{\mu_1^2} + \frac{E[\zeta_t^2]}{\mu_1^2}$$

$$+ \left[\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1}\right](T-E\zeta_t) + 1 - \frac{3\mu_2}{2\mu_1^2} + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3}$$

$$+ \frac{3\mu_2}{2\mu_1^2}[1 - F_{(2)}*G_t(T)] - \frac{3\mu_2^2}{2\mu_1^4}[1 - F_{(2)}*F_{(2)}*G_t(T)]$$

$$+ \frac{2\mu_3}{3\mu_1^3}[1 - F_{(3)}*G_t(T)] + K*G_t(T) - 3L*G_t(T).$$

After applying Theorem 6.1 the above expressions can be combined and simplified to yield estimates for $E\{N(t;\ T)\}$ and $\mathrm{Var}\{N(t;\ T)\}$. We omit the details, since the computation involved is awkward but straightforward. It is, however, worth mentioning that terms such as $1 - F_{(2)}{*}G_t(T)$ are of the order of magnitude $o(1/TM(t))$ as $T \to \infty$ *uniformly with respect to* t; this is not difficult to show with the aid of the following lemma:

LEMMA 6.3 *If* $F(x) \in \mathcal{D}(M;\ 3)_\cap C$ *for some* $M(x) \in M^*$, *the integral*

$$\int_0^\infty xM(x)\,dF_{(2)}{*}G_t(x)$$

converges uniformly with respect to t.

PROOF. Since $M(x) \in M^*$, there exists a positive constant C such that

$$vM(v) \le C\int_0^v M(u)\,du, \qquad (v \ge 0).$$

Furthermore, since $F_{(2)}(x)$ possesses a density $f_{(2)}(x)$, the convolution $F_{(2)}{*}G_t(x)$ is absolutely continuous with density

$$\int_0^x f_{(2)}(x-v)\,dG_t(v).$$

By Theorem 2.1

$$A = \int_0^\infty wM(w)f_{(2)}(w)\,dw < \infty.$$

Using Fubini's theorem, for fixed $T > 0$,

$$\int_0^T xM(x)\int_0^x f_{(2)}(x-v)dG_t(v)dx = \int_0^T\int_0^T xM(x)f_{(2)}(x-v)dxdG_t(v) \tag{6.2.3}$$

$$= \int_0^T\int_0^{T-v}(v+w)M(v+w)f_{(2)}(w)dwdG_t(v)$$

$$\leq \int_0^T vM(v)\int_0^{T-v} M(w)f_{(2)}(w)dwdG_t(v)$$

$$+ \int_0^T M(v)\int_0^{T-v} wM(w)f_{(2)}(w)dwdG_t(v)$$

$$\leq 2AC\int_0^T\int_0^v M(u)dudG_t(v) = 2AC\int_0^T M(u)\int_0^T dG_t(v)du$$

$$\leq 2AC\int_0^T M(u)[1 - G_t(u)]du \ .$$

By Lemma 6.2 and Fubini's theorem,

$$2AC\int_0^T M(u)[1 - G_t(u)]du \leq 2AC\int_0^T M(u)[1 - D(u)]du$$

$$\leq 2AC\int_0^\infty uM(u)dD(u) < \infty.$$

The lemma follows by taking the limit as $T \to \infty$ of the left-hand side of (6.2.3).

As a consequence of Lemma 6.3,

$$TM(T)[1 - F_{(2)}*G_t(T)] \leq \int_T^\infty xM(x)dF_{(2)}*G_t(x) \to 0$$

uniformly in t as $T \to \infty$, so that

$$1 - F_{(2)}*G_t(T) = o[\frac{1}{TM(T)}] \ .$$

We have now obtained the following result:

THEOREM 6.4 *Suppose* $F(x) \in \mathcal{D}(M; 2)_\cap C$ *for some* $M(x) \in M^*$ *and that* t *is fixed but moderately large. Then as* $T \to \infty$,

$$E\{N(t; T)\} = \frac{T}{\mu_1} + \frac{\mu_2}{2\mu_1^2} + o\left[\frac{1}{tM(t)}\right]$$

$$- \frac{\mu_2^2}{2\mu_1^2}\left[1 - F_{(2)}*G_t(T)\right] + o\left[\frac{1}{TM(T)}\right] .$$

If, in addition, $F(x) \in \mathcal{D}(M; 3)_\cap C$, *then as* $T \to \infty$,

$$\mathrm{Var}\{N(t; T)\} = T\left[\frac{\mu_2 - \mu_1^2}{\mu_1^3} + o\left[\frac{1}{tM(t)}\right]\right]$$

$$+ \frac{5\mu_2^2}{4\mu_1^4} - \frac{\mu_3}{\mu_1^3} - \frac{\mu_2}{2\mu_1^2} + o\left[\frac{1}{tM(t)}\right]$$

$$+ \frac{\mu_2 t}{\mu_1^3}\left[1 - F_{(2)}(t)\right] + \frac{\mu_2}{\mu_1^3}\int_o^t u\,dF_{(2)}(u)$$

$$+ \frac{2\mu_3}{3\mu_1^3}\left[1 - F_{(3)}*G_t(T)\right] + \frac{\mu_2 T}{\mu_1^3}\left[1 - F_{(2)}*G_t(T)\right]$$

$$+ o\left[\frac{1}{TM(T)}\right] .$$

Using Theorem 6.4 we can estimate the covariance of the numbers of renewals in contiguous intervals:

COROLLARY 6.5 *Suppose* $F(x) \in \mathcal{D}(M;\ 3)_{\cap} C$ *for some* $M(x) \in M^*$ *and that* $N(0, T_1)$ *and* $N(T_1, T_2)$ *are the numbers of renewals in the intervals* $(0, T_1]$ *and* $(T_1, T_1+T_2]$, *respectively, where* T_1 *is fixed with respect to* T_2. *Then*

$$\text{(6.2.4)}\quad \mathrm{Cov}(N(0,T_1),\ N(T_1,T_2)) = o\left[\frac{1}{T_1M(T_1)}\right]T_2 + \frac{-5u_2^2}{8\mu_1^4} + \frac{\mu_3}{2\mu_1^3} + \frac{\mu_2}{4\mu_1^2}$$

$$+ o\left[\frac{1}{T_1M(T_1)}\right] - \frac{\mu_2T}{\mu_1^3}\,[1 - F_{(2)}(T_1)] - \frac{\mu_2}{2\mu_1^3}\int_o^{T_1} u\,dF_{(2)}(u)$$

$$- \frac{\mu_3}{3\mu_1^3}\,[1 - F_{(3)}(T_1)] + o\left[\frac{1}{T_1M(T_1)}\right]$$

$$+ \frac{\mu_2(T_1+T_2)}{2\mu_1^3}\,[1 - F_{(2)}(T_1+T_2)] + \frac{\mu_3}{3\mu_1^3}\,[1 - F_{(3)}(T_1+T_2)]$$

$$- \frac{\mu_3}{3\mu_1^3}\,[1 - F_{(3)}{*}G_{T_1}(T_2)] - \frac{\mu_2T_2}{2\mu_1^3}\,[1 - F_{(2)}{*}G_{T_1}(T_2)]$$

$$+ o\left[\frac{1}{T_2M(T_2)}\right].$$

PROOF. If X and Y are random variables, then

$$\text{(6.2.5)}\qquad 2\,\mathrm{Cov}(X,Y) = \mathrm{Var}(X+Y) - \mathrm{Var}\ X - \mathrm{Var}\ Y\ .$$

Theorems 6.4 and 5.8, together with this identity, imply the corollary.

Corollary 6.5 has an immediate application to the superposition of renewal processes. Suppose that (as described in Chapter 3) N iid renewal processes are superposed and that $N^*(0, T_1)$ and $N^*(T_1, T_2)$ denote the numbers of events occurring during the time intervals $(0, T_1]$ and $(T_1, T_2]$, respectively, *for the superposition.* Then $\frac{1}{N}\,\mathrm{Cov}(N^*(0, T_1), N^*(T_1, T_2))$ is given by (6.2.4).

Finally we note that the identity (6.2.5) can easily be extended to sums of more than two random variables. Theorems 6.4 and 5.8 can then be used in a straightforward fashion to estimate the covariance of the numbers of renewals in non-adjacent time intervals.

APPENDIX: PROOF OF THEOREM 2.2

W.L. Smith (1967) proved a version of the Wiener-Pitt-Lévy result in which it is assumed that the moment function $M(x)$ belongs to the somewhat restrictive class M^*; see Theorem 2 of Smith (1967; page 270). Later in a study of the theory of recurrent events, Smith (1976) introduced the broader class M and extended his earlier theorem to include moment functions $M(x)$ in M; see Theorem 3.1 of Smith (1976; page 41). Our Theorem 2.2 differs from the former only in that we have used the class M rather than M^*. However, since Theorem 3.1 of Smith (1976) refers to Fourier series rather than Fourier-Stieltjes transforms, we have found it necessary to make some non-trivial changes in the proof of Smith's earlier result. Fortunately several technical lemmas needed to carry out the modifications are given in Section 3 of the study by Smith (1976), and we shall simply quote these without proof.

The essential tool used by Smith (1967) to prove Theorem 2 is the smooth mutilator function (SMF) written as $q^{\dagger}(\theta;\ \alpha,\beta,\gamma,\delta)$. However the SMF developed by Smith in his 1967 paper is inadequate when $M(x) \in M$, and consequently Smith (1976) later constructed a refined smooth mutilator function. This new SMF has the properties described in Section 2.1, and, furthermore,

$$q(x;\ \alpha,\beta,\gamma,\delta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\, q^{\dagger}(\theta;\ \alpha,\beta,\gamma,\delta)\,d\theta$$

belongs to the class $L(M; \nu)$ *for any* $M(x) \in M$ *and any* $\nu \geq 0$. The construction given by Smith (1976; pages 41-48) is entirely adequate for our purposes, and we refer the reader to that paper for the details in order to avoid duplicating them here.

We first prove Part A of Theorem 2.2. (For convenience we shall use the notation of Smith (1967, pages 283-287).) Let θ_o be any fixed point in the compact interval J (possible an end-point). Then $\Phi(z)$ is analytic at $z = \phi(\theta_o)$ and therefore has the Taylor expansion

$$\Phi(z) = \Phi(\phi(\theta_o)) + \sum_{n=1}^{\infty} c_n(z - \phi(\theta_o))^n$$

about the point $z = \phi(\theta_o)$ with a strictly positive radius of convergence (say) ρ_o. Since $\phi(\theta)$ is continuous, there exists a constant $\delta_o > 0$ such that $|\phi(\theta) - \phi(\theta_o)| \leq \rho_o$ for all θ such that $|\theta-\theta_o| \leq \delta_o$. Thus for $|\theta-\theta_o| \leq \delta_o$,

$$\Phi(\phi(\theta_o)) = \Phi(\phi(\theta_o)) + \sum_{n=1}^{\infty} c_n(\phi(\theta) - \phi(\theta_o))^n .$$

Write $q^{\dagger}(\theta)$ for the special SMF $q^{\dagger}(\theta; -2, -1, 1, 2)$ and for fixed $\lambda > 0$ define

$$T_{\lambda}^{\dagger}(\theta) = q^{\dagger}\left(\frac{\theta-\theta_o}{\lambda/2}\right) \Phi(\phi(\theta)).$$

Then $T_{\lambda}^{\dagger}(\theta)$ and $\Phi(\phi(\theta))$ are identical if $|\theta-\theta_o|$ is sufficiently small. Since

$$q^{\dagger}\left(\frac{\theta-\theta_o}{\lambda/2}\right) = q^{\dagger}\left(\frac{\theta-\theta_o}{\lambda/2}\right) \left[q^{\dagger}\left(\frac{\theta-\theta_o}{\lambda}\right)\right]^n$$

for any $n > 0$, it follows that

$$T_\lambda^\dagger(\theta) = \Phi(\phi(\theta_o))\, q^\dagger\left(\frac{\theta-\theta_o}{\lambda/2}\right)$$

$$+ \sum_{n=1}^{\infty} c_n\, q^\dagger\left(\frac{\theta-\theta_o}{\lambda/2}\right)\left[q^\dagger\left(\frac{\theta-\theta_o}{\lambda}\right)\left\{\phi(\theta) - \phi(\theta_o)\right\}\right]^n .$$

Clearly the function

$$q^\dagger\left(\frac{\theta-\theta_o}{\lambda}\right)\left\{\phi(\theta) - \phi(\theta_o)\right\}$$

is the Fourier transform of $r_\lambda(x)$, say, where

$$r_\lambda(x) = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta_o(x-z)}\{q(\lambda(x-z)) - q(\lambda x)\}dB(z),$$

and $B(x)$ is the function in $\mathcal{B}(M;\nu)$ for which $\phi(\theta) = B^\ddagger(\theta)$.

By Fubini's theorem for nonnegative functions

$$\int_{-\infty}^{\infty}|r_\lambda(x)|dx \le \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|q(\lambda(x-z)) - q(\lambda x)|dx\right\}|dB(z)|$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}|q(u-\lambda z) - q(u)|du\right\}|dB(z)| .$$

The inner integral

$$\int_{-\infty}^{\infty}|q(u-\lambda z) - q(u)|du$$

tends to zero boundedly as $\lambda \to 0$. Therefore by choosing λ sufficiently small we may suppose

$$\int_{-\infty}^{\infty} |r_\lambda(x)|dx < \rho_o .$$

Since $r_\lambda^\dagger(\theta)$ is the product of a function in $B^\ddagger(M;\nu)$ and an SMF, it follows that $r_\lambda(x) \in L(M;\nu)$. Now set

$$p^\dagger(\theta) = \sum_{n=1}^{\infty} c_n \left[q^\dagger\left(\frac{\theta-\theta_o}{\lambda}\right)\left\{\phi(\theta) - \phi(\theta_o)\right\}\right]^n$$

$$= \sum_{n=1}^{\infty} c_n \left[r_\lambda^\dagger(\theta)\right]^n .$$

Then

$$p(x) = \sum_{n=1}^{\infty} c_n\, r_\lambda^{*n}(x),$$

where the convolutions of $r_\lambda(x)$ are defined as follows:

$$r_\lambda^{*1}(x) = r_\lambda(x)$$

and $$r_\lambda^{*n}(x) = \int_{-\infty}^{\infty} r_\lambda(x-z)\, r_\lambda^{*(n-1)}(z)dz, \quad (n = 2,3,\ldots).$$

In order to show that $p(x) \in L(M;\nu)$ for $M(x) \in M$ we make use of the following result:

<u>LEMMA A.1</u> *If* $\Psi(z)$ *is analytic in the disc* $|z| \le r$, *if* $a(x) \in L(M;\nu)$ *for some* $M(x) \in M$ *and some* $\nu \ge 0$, *and if*

$$\int_{-\infty}^{\infty} |a(x)|dx < r,$$

then $\Psi(a(x)) \in L(M;\nu)$.

PROOF OF LEMMA A.1. The proof of Lemma A.1 is based on two auxiliary lemmas given by Smith (1976):

LEMMA A.2 *Suppose* $a(x)$ *and* $b(x)$ *belong to* $L(M;\nu)$ *for some right moment function* $M(x)$ *and some* $\nu \ge 0$. *Then*

$$\int_{-\infty}^{\infty} M(x)x^{\nu}\,|a(x)*b(x)|dx \le$$

$$\left\{\int_{-\infty}^{\infty} M(x)x^{\nu}|a(x)|dx\right\}\left\{\int_{-\infty}^{\infty} M(x)x^{\nu}\,|b(x)|dx\right\}.$$

PROOF. See Smith (1976; page 21).

LEMMA A.3 *If* $a(x) \in L(M;\nu)$ *where* $M(x)$ *is a right moment function and* $\nu \ge 0$, *then there exists a function* $\varepsilon(x)$ *such that* $\varepsilon(x) \to 0$ *as* $x \to \infty$ *and*

$$\int_{-\infty}^{\infty} M(x)x^{\nu}|a^{*k}(x)|dx \le \left[\int_{-\infty}^{\infty}|a(x)|dx + \varepsilon(x)\right]^{k}$$

for all $k \ge 1$.

PROOF. See Smith (1976; pages 21-22).

We can find $\varepsilon > 0$ such that

$$\int_{-\infty}^{\infty}|a(x)|dx + \varepsilon < r,$$

and by Lemma A.3 there exists a positive integer k_0 (depending on ε) such that

$$\int_{-\infty}^{\infty} M(x)x^{\nu} \, |a^{*m}(x)|\,dx \le \left\{\int_{-\infty}^{\infty} |a(x)|\,dx + \varepsilon\right\}^m$$

for all $m \ge k_o$. Thus if

$$\Psi(a) = \sum_{m=0}^{\infty} q_m z^m \quad \text{(say)},$$

and we define

$$\Psi_{k_o}(a(x)) = \sum_{m=k_o}^{\infty} q_m \, a^{*m}(x),$$

then

$$\int_{-\infty}^{\infty} M(x)x^{\nu}|\Psi_{k_o}(a(x))|\,dx \le$$

$$\le \sum_{m=k_o}^{\infty} |q_m| \left\{ \int_{-\infty}^{\infty} |a(x)|\,dx + \varepsilon\right\}^m < \infty,$$

since the series $\sum_{m=0}^{\infty} q_m z^m$ is absolutely convergent within its radius of convergence. But

$$\int_{-\infty}^{\infty} M(x)x^{\nu}|\Psi(a(x)) - \Psi_{k_o}(a(x))|\,dx$$

$$\le \sum_{m=0}^{k_o-1} |q_m| \int_{-\infty}^{\infty} M(x)x^{\nu} \, |a^{*m}(x)|\,dx < \infty$$

by finitely many applications of Lemma A.2. Therefore we can conclude that $\Psi(a(x)) \in L(M; \nu)$, thus proving Lemma A.1.

Now the Fourier transform of $p(x)$ is given by

$$p^{\dagger}(\theta) = \sum_{n=1}^{\infty} c_n \left[q^{\dagger}\left(\frac{\theta-\theta_o}{\lambda}\right) \left\{\phi(\theta) - \phi(\theta_o)\right\}\right]^n ,$$

and it follows from Lemma A.2 that $T_{\lambda}^{\dagger}(\theta) \in L^{\dagger}(M;\ \nu)$.

So far we have shown that if θ_o is any point of the closed interval J (including the end-points), then there exists a function $T_{\lambda}^{\dagger}(\theta) \in L^{\dagger}(M;\ \nu)$ such that $\Phi(\phi(\theta)) = T_{\lambda}^{\dagger}(\theta)$ for all θ in a closed sub-interval centered about θ_o. By the Heine-Borel theorem it is possible to cover the closed bounded interval J with a finite number of these intervals.

Suppose $(\beta_1,\ \gamma_1)$ and $(\beta_2,\ \gamma_2)$ are two such overlapping intervals, so that $\beta_1 < \beta_2 < \gamma_1 < \gamma_2$, and suppose that $T_{\lambda_1}^{\dagger}(\theta)$ and $T_{\lambda_2}^{\dagger}(\theta)$ are two corresponding functions in $L^{\dagger}(M;\ \nu)$. Then

$$q^{\dagger}(\theta;\ \beta_1-1,\beta_1,\beta_2,\gamma_1)\ T_{\lambda_1}^{\dagger}(\theta) + q^{\dagger}(\theta;\ \beta_2,\gamma_1,\gamma_2,\gamma_2+1)T_{\lambda_2}^{\dagger}(\theta)$$

belongs to $L^{\dagger}(M;\ \nu)$ and is identically equal to $\Phi(\phi(\theta))$ throughout $(\beta_1,\ \gamma_2)$. This argument may be continued in an obvious manner (with appropriate modifications at the end-points of J,) so that we obtain a single function $T^{\dagger}(\theta)$, say, which belongs to $L^{\dagger}(M;\ \nu)$ and is identically equal to $\Phi(\phi(\theta))$ for $\theta \in J$. In fact, since $\psi(\theta)$ vanishes outside J, we have

$$\psi(\theta)T^{\dagger}(\theta) = \psi(\theta)\Phi(\phi(\theta)) \quad \text{for all } \theta,$$

and since $\psi(\theta)T^{\dagger}(\theta)$ belongs to $L^{\dagger}(M;\ \nu)$ by Lemma A.2, we have

$\psi(\theta)\Phi(\phi(\theta)) \in L^{\dagger}(M;\ \nu)$. This implies the conclusion in the first part of Theorem 2.2.

The proof of Part B involves the use of a result analogous to Lemma A.1 for functions of bounded variation. Since this additional lemma can be established in a straightforward manner, we shall omit the details for the sake of brevity and merely outline the second half of the proof of Theorem 2.2.

Suppose (without loss of generality) that $J = (\eta, +\infty)$, and write ρ for $\rho[\phi(\theta)]$. Since no singularity of $\Phi(z)$ is within a distance ρ of the origin by assumption, we can find an $\varepsilon > 0$ such that $\Phi(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $|z| < \rho + 2\varepsilon$, and the series converges absolutely in this interval. There exists a positive integer k such that

$$\{\phi(\theta)\}^k = \alpha a^{\dagger}(\theta) + \beta B^{\ddagger}(\theta),$$

where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$, $a^{\dagger}(\theta) \in L^{\dagger}_{\eta}\, \mathcal{D}^{\ddagger}(M;\ \nu)$, $B^{\ddagger}(\theta) \in \mathcal{D}^{\ddagger}(M;\ \nu)$, and $\beta < (\rho + \frac{1}{2}\varepsilon)^k$. (The fact that both α and β are real is a consequence of assuming that $\phi(\theta)$ is a characteristic function.) For all $|\theta| > 2\lambda$, say, $|\phi(\theta)| < \rho + \varepsilon$ and therefore

$$\begin{aligned}
\Phi(\phi(\theta)) &= \sum_{n=0}^{\infty} c_n (\phi(\theta))^n \\
&= \sum_{j=0}^{k-1} \sum_{n=0}^{\infty} c_{nk+j} (\phi(\theta))^{nk+j} \\
&= \sum_{j=0}^{k-1} (\phi(\theta))^j \sum_{n=0}^{\infty} c_{nk+j} (\alpha a^{\dagger}(\theta) + \beta B^{\ddagger}(\theta))^n \\
&= \sum_{j=0}^{k-1} (\phi(\theta))^j\, \Xi_j(\theta), \quad \text{say.}
\end{aligned}$$

For $\theta \geq 2\lambda$,

$$\Xi_j(\theta) = \sum_{n=0}^{\infty} c_{nk+j}(\alpha a^{\dagger}(\theta) + \beta B^{\ddagger}(\theta) - \alpha a^{\dagger}(\theta) q^{\dagger}(\tfrac{\theta}{\lambda}))^n ,$$

and this series converges absolutely. Since $a^{\dagger}(\theta)$ is the Fourier transform of the $L(M; \nu)$-function $a(x)$, it follows that $a^{\dagger}(\theta) - a^{\dagger}(\theta)q^{\dagger}(\theta/\lambda)$ is the Fourier transform of the $L(M; \nu)$-function $g_{\lambda}(x)$, say, where

$$g_{\lambda}(x) = \int_{-\infty}^{\infty} [a(x) - a(x - \tfrac{u}{\lambda})]\, q(u)du,$$

using the fact that $\int_{-\infty}^{\infty} q(u)du = q^{\dagger}(0) = 1$. By arguments similar to those used in the proof of Part A, we can show that

$$\int_{-\infty}^{\infty} |g_{\lambda}(x)|dx \to 0 \qquad \text{as} \quad \lambda \to \infty.$$

Thus for λ sufficiently large, $\beta B^{\ddagger}(\theta) + \alpha g_{\lambda}^{\dagger}(\theta)$ is a function of $B^{\ddagger}(M; \nu)$ whose total variation is less than $(\rho + \varepsilon)^k$. As before, it follows that $\Xi_j(\theta) \in B^{\ddagger}(M; \nu)$, since $\Phi(z)$ is analytic in $|z| < \rho + 2\varepsilon$.

Since $\{\phi(\theta)\}^j \in B^{\ddagger}(M; \nu)$ for every integer $j \geq 0$ by Lemma A.2, it follows that there is some function $T_{\infty}^{\ddagger}(\theta)$, say, belonging to $B^{\ddagger}(M; \nu)$ and such that

$$[1 - q^{\dagger}(\theta/2\lambda)]\, \psi(\theta)\Phi(\phi(\theta)) = T_{\infty}^{\ddagger}(\theta)$$

for all θ. By the first part of the proof there is also a function $T_{o}^{\ddagger}(\theta)$ belonging to $B^{\ddagger}(M; \nu)$ and such that

$$q^{\dagger}(\theta/2\lambda)\ \psi(\theta)\Phi(\phi(\theta)) = T_o^{\ddagger}(\theta)$$

for all θ. Thus

$$\psi(\theta)\Phi(\phi(\theta)) = T_o^{\ddagger}(\theta) + T_{\infty}^{\ddagger}(\theta) \in B^{\ddagger}(M;\ \nu),$$

which concludes the proof of Part B of the theorem.

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A so-called *moment class* of non-decreasing functions M(x) is defined and for a real constant $\nu\leq 0$ the class of functions $B(M;\nu)$ consists of all functions of bounded variation such that $\int_{-\infty}^{+\infty}|x|^{\nu}M(|x|)|dB(x)| < \infty$ exists. Renewal processes are considered and it is supposed the lifetime distribution function belongs to a class $B(M;\nu)$; approximations are then obtained to various moments of the renewal count N(t), the remainder terms bring in a related B-class. These results

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[20] are used to tackle various questions concerned with superposed renewal processes.